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Mock modularity of the M_d-rank of overpartitions

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ABSTRACT

We investigate the modular properties of a new partition rank, the M_d-rank of overpartitions. In fact this is an infinite family of ranks, indexed by the positive integer d, that gives both the Dyson rank of overpartitions and the overpartition M_2-rank as special cases. The M_d-rank of overpartitions is the holomorphic part of a certain harmonic Maass form of weight $\frac{1}{2}$. We give the exact transformation of this harmonic Maass form along with a few identities for the M_d-rank.

1. Introduction

The theory of mock modular forms began with the introduction of mock theta functions in Ramanujan's last letter to Hardy. Currently we understand mock theta functions as special cases of mock modular forms, which we in turn recognize as the holomorphic parts of harmonic Maass forms. Harmonic Maass forms are most easily understood as classical half-integer weight modular forms with relaxed analytic conditions. With the monumental thesis of Zwegers [28], we can now often take functions we expect to be mock modular, write them in terms of basic building block functions, and complete them to harmonic Maass forms. We can then work with these harmonic Maass forms in terms of their building blocks with ease, comparable to working with classical modular forms in terms of Dedekind's η-function and modular units.

Inherent to studying mock theta functions is the theory of partitions. A partition of a nonnegative integer n is a nonincreasing sequence of positive integers, called parts, that sum to n. We let $p(n)$ denote the number of partitions of n. As an example, $p(4) = 5$ as the partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$. This definition is deceitful in its simplicity, as illustrated by the fact that of the following two series, one is the generating function for $p(n)$ and the other is a third order mock theta function of Ramanujan,

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$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2(1-q^3)^2\cdots(1-q^n)^2}, \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2(1+q^3)^2\cdots(1+q^n)^2}.$$

We can recognize both of these series as special cases of the function $\widetilde{R}(z;q)$ given by

$$\widetilde{R}(z;q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-zq)(1-q/z)(1-zq^2)(1-q^2/z)\cdots(1-zq^n)(1-q^n/z)}.$$

Other choices of z give other third order mock theta functions, so we may call $\widetilde{R}(z;q)$ a universal mock theta function, but it also has another name. The function $\widetilde{R}(z;q)$ is actually the generating function for the rank of partitions and was introduced by Dyson [12]. The rank of a partition is given by taking the largest part of the partition and subtracting off the number of parts of the partition. Now we see partitions and mock modular forms are somehow linked.

Another partition function is the overpartition function. An overpartition of n is a partition of n in which the first appearance of a part may be overlined. For example the overpartitions of 3 are 3, $\overline{3}$, $2+1$, $2+\overline{1}$, $\overline{2}+1$, $\overline{2}+\overline{1}$, $1+1+1$, and $\overline{1}+1+1$. We let $\overline{p}(n)$ denote the number of overpartitions of n. The previous example shows that $\overline{p}(3) = 8$. It turns out this function also fits well into the theory of q-series and modular forms. It particular the generating function is the rather elegant η-quotient $\frac{\eta(2\tau)}{\eta(\tau)^2}$ and the inverse of this function generates the sequence of square numbers. This can be compared with the fact that the generating function for $p(n)$ is $\frac{1}{\eta(\tau)}$ and the inverse generates the sequence of pentagonal numbers. Overpartitions also have ranks associated to them. There is the Dyson rank of an overpartition, which ignores whether or not a part is overlined and is just the largest part minus the number of parts. There is also the M_2-rank of overpartitions, which is given by

$$\left\lceil \frac{l(\pi)}{2} \right\rceil - \#(\pi) + \#(\pi_o) - \chi(\pi),$$

where $l(\pi)$ is the largest part of π, $\#(\pi)$ is the number of parts, $\#(\pi_o)$ is the number of odd non-overlined parts, and $\chi(\pi) = 1$ when the largest part is odd and non-overlined and $\chi(\pi) = 0$ otherwise. Both of the generating functions of these ranks can also be considered as universal mock theta functions, as specializations yield known mock theta functions.

In this article we consider a family of rank-type generating functions for overpartitions, which generalize both the Dyson rank and the M_2-rank. Generalizations and families of ranks is not unheard of, for example there is Garvan's k-rank [13], which is defined by the series

$$\frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{\frac{n((2k-1)n-1)}{2}}(1-q^n)(1-q^{2n})}{(1-zq^n)(1-z^{-1}q^n)},$$

and whose combinatorial interpretation is related to successive Durfee squares. Also in [2] Andrews introduced Durfee symbols and k-marked Durfee symbols, yielding "Dyson-like" ranks.

The function we consider is the M_d-rank of overpartitions, which is defined by the series

$$\begin{aligned}\mathcal{O}_d(z;q) &= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M_d(m,n) z^m q^n \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty} \left(1 + 2\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+dn}}{(1-zq^{dn})(1-z^{-1}q^{dn})}\right), \qquad (1.1)\end{aligned}$$