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Dispersion relations in hot magnetized plasmas

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A B S T R A C T

In the framework of hot magnetized collisionless plasmas, dispersion relations have been extensively studied in the past [2,12,13,24,33,34,38]. This subject is still topical in plasma physics [19,27,32,36,42]. The aim of this article is to provide a rigorous derivation of the characteristic variety, based on some asymptotic analysis of the relativistic Vlasov–Maxwell system. Special emphasis is made on the modeling of Tokamaks, with spatial variations of the magnetic field and of the equilibrium distribution function. In order to take into account the inhomogeneities, the problem is formulated in terms of geometrical optics [29,31]. This allows to unify, justify and extend the preceding results. New aspects are indeed included. For instance, the dielectric tensor is defined for real frequencies through singular integrals involving the Hilbert transform.

Contents

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1. Introduction

The dispersion relations have been extensively studied in plasma physics. It is because they are involved in a wide range of astrophysical contexts and laboratory experiments through wave-particle interaction [23,39], transfer of power between waves and particles, heating of plasmas [18], reflectometry techniques [11,21], and so on. The preparatory works from the 1960s, 1970s and 1980s [2,12,13,24,33,34,38] are the template for recent numerical studies [36,42], for contemporary investigations in more complex situations [19,27,30,32] or, like in the present text which is about tokamaks, for developments up to the case of *non-uniform* magnetized plasmas.

In real fusion machines, the dominant distribution function and the external magnetic field are inhomogeneous. They undergo significant fluctuations in position. These variations have a major effect on the geometry of wave propagation. Their impact is important when performing ray tracing, with many practical consequences. It becomes decisive when looking at the transport equations (to measure power transfers between waves and particles) or in the perspective of long time studies [5,6]. However, the presence of inhomogeneities is complicated to simulate. This is probably why, despite some attempts [35,37], this subject has not been completely studied. Another reason is, without a doubt, a general principle of physics according to which a dispersion relation can be obtained by analyzing a plane monochromatic wave in a homogeneous medium, and then letting the medium's properties (in the dielectric tensor) vary slowly in position. After verification, this principle holds true, but it is not so easy to determine what should vary in the dispersion relation, why and how. There are questions that remain unanswered. The aim of this article is precisely to check what the situation really is. It is to rigorously define the characteristic variety by extracting the corresponding dielectric tensor through a comprehensive study. To this end, it is not enough to extend existing procedures, which give formal results, provide partial information or rely on specific hypotheses. A new approach is needed.

In a plasma, the presence of a strong magnetic field makes the electrons oscillate at the electron cyclotron frequency ε^{-1} with $\varepsilon \ll 1$. Away from thermal equilibrium, the repartition of the electrons is therefore described by oscillating kinetic distribution functions whose structures are exhibited in [6]. This produces oscillating currents. Then, by a mesoscopic caustic effect [5], self-consistent oscillating electromagnetic waves are emitted. They act like coherent sources [7]. Roughly speaking, it is as if the rays emanate from a *smooth nonlinear phase* $\phi(\mathbf{t},\mathbf{x})$. The same applies to waves launching by antennas, in view of the radio frequency heating of tokamak plasmas.

It turns out that the propagation of electromagnetic oscillations in a hot quasi-neutral background of ions and electrons can be described in the framework of some asymptotic analysis. To some extent, we can consider WKB expansions involving a single phase $\phi(\mathbf{t},\mathbf{x})$, as in (3.3). From there, the matter is to construct for the relativistic Vlasov–Maxwell system an adequate geometrical optics. In comparison with usual theories in hyperbolic equations [29,31], new difficulties come from the kinetic resonances which are hidden in the self-consistent picture.

As a matter of fact, the propagation of waves is still governed by a dielectric tensor $\sigma(\cdot)$. But now the dielectric property becomes a reactive aspect of the wave-particle interaction. The aim of this article is to derive $\sigma(\cdot)$ from basic principles. Then, it is to rigorously define the content of $\sigma(\cdot)$ in the domain of *real* frequencies. When doing this, complications arise for instance from the singular integrals that play a part in the construction of $\sigma(\cdot)$.

Theorem 1 *(Eikonal equation in axisymmetric configurations). There exists a well-defined skew-symmetric matrix $\sigma(\cdot)$ playing the part of a conductivity tensor, such that the eikonal equation governing wave propagation in tokamaks can be determined through the following Hamilton–Jacobi equation:*

$$\det\left(\nabla_{\mathbf{x}}\phi\,{}^t\nabla_{\mathbf{x}}\phi + (\partial_{\mathbf{t}}\phi)^2\,Id - |\nabla_{\mathbf{x}}\phi|^2\,Id + i\,\partial_{\mathbf{t}}\phi\;\sigma(\mathbf{x},\partial_{\mathbf{t}}\phi,\nabla_{\mathbf{x}}\phi)\right) = 0\,. \tag{1.1}$$

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