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A priori estimates for the Fitzpatrick function

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To the memory of Jerome S. Cohen (1941–2018)

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ABSTRACT

New perspectives, proofs, and some extensions of known results are presented concerning the behavior of the Fitzpatrick function of a monotone type operator in the general context of a locally convex space.

1. Introduction and preliminaries

Given $T : X \rightrightarrows X^*$ a multi-valued operator defined in a locally convex space X with valued in its topological dual X^*, the Fitzpatrick function associated to T (introduced in [1, Definition 3.1, p. 61]) denoted by

$$\varphi_T(x, x^*) := \sup\{a^*(x - a) + x^*(a) \mid (a, a^*) \in \operatorname{Graph} T\},\ (x, x^*) \in X \times X^*,$$

is an important tool in the theory of maximal monotone operators (see e.g. [1,6,11]).

Here $\operatorname{Graph} T = \{(x, x^*) \in X \times X^* \mid x^* \in T(x)\}$ is the graph of T, $D(T) = \operatorname{Pr}_X(\operatorname{Graph} T)$ stands for the domain of T, $R(T) = \operatorname{Pr}_{X^*}(\operatorname{Graph} T) = \cup_{x\in D(T)}T(x)$ denotes the range of T, where Pr_X, Pr_{X^*} are the projections of $X \times X^*$ onto X, X^*, respectively. When no confusion can occur, $T : X \rightrightarrows X^*$ is identified with $\operatorname{Graph} T \subset X \times X^*$.

Our focus in this note is on the general behavior of φ_T mainly with respect to the coupling of $X \times X^*$ as well as the position of the domain of φ_T in $X \times X^*$.

In this paper, if not otherwise explicitly mentioned, (X, τ) is a Hausdorff separated locally convex space (LCS for short), X^* is its topological dual endowed with the weak-star topology w^*, and the topological dual of (X^*, w^*) is identified with X. For $x \in X$ and $x^* \in X^*$ we set $\langle x, x^*\rangle := x^*(x)$.

For a subset A of X we denote by $\operatorname{int} A$, $\operatorname{cl} A$ (or $\operatorname{cl}_\tau A$ when we wish to underline the topology τ, or $\operatorname{cl}_{(X,\tau)} A$ if we want to emphasize that $A \subset (X, \tau)$), $\operatorname{aff} A$, and $\operatorname{conv} A$ the interior, the closure, the affine

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hull, and the convex hull of A, respectively. If $A, B \subset X$ we set $A + B := \{a + b \mid a \in A,\ b \in B\}$ with the convention $A + \emptyset := \emptyset + A := \emptyset$.

We consider the class $\Lambda(X)$ of proper convex functions $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and the class $\Gamma_\tau(X)$ (or simply $\Gamma(X)$) of those functions $f \in \Lambda(X)$ which are τ-lower semicontinuous (lsc for short). Recall that f is *proper* if $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$ is nonempty and f does not take the value $-\infty$.

To $f : X \to \overline{\mathbb{R}}$ we associate its *convex hull* $\operatorname{conv} f : X \to \overline{\mathbb{R}}$ and its (τ-)*lsc convex hull* $\operatorname{cl}\operatorname{conv} f : X \to \overline{\mathbb{R}}$ ($\operatorname{cl}_\tau \operatorname{conv} f$ when we want to accentuate on the topology τ) defined by

$$(\operatorname{conv} f)(x) := \inf\{t \in \mathbb{R} \mid (x, t) \in \operatorname{conv}(\operatorname{epi} f)\},$$
$$(\operatorname{cl}\operatorname{conv} f)(x) := \inf\{t \in \mathbb{R} \mid (x, t) \in \operatorname{cl}\operatorname{conv}(\operatorname{epi} f)\},$$

where $\operatorname{epi} f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \leq t\}$ is the *epigraph* of f.

The *conjugate* of $f : X \to \overline{\mathbb{R}}$ with respect to the dual system (X, X^*) is given by

$$f^* : X^* \to \overline{\mathbb{R}}, \quad f^*(x^*) := \sup\{\langle x, x^* \rangle - f(x) \mid x \in X\}. \tag{1}$$

The conjugate f^* is a weakly-star (or w^*-) lsc convex function. For the proper function $f : X \to \overline{\mathbb{R}}$ we define the *subdifferential* of f at x by

$$\partial f(x) := \{x^* \in X^* \mid \langle x' - x, x^* \rangle \leq f(x') - f(x)\ \forall x' \in X\},$$

for $x \in \operatorname{dom} f$ and $\partial f(x) := \emptyset$ for $x \notin \operatorname{dom} f$. Recall that $N_C = \partial \iota_C$ is the *normal cone* of C, where ι_C is the *indicator function* of $C \subset X$ defined by $\iota_C(x) := 0$ for $x \in C$ and $\iota_C(x) := \infty$ for $x \in X \setminus C$.

When X^* is endowed with the topology w^* (or with any other locally convex topology σ such that $(X^*, \sigma)^* = X$), in other words, if we take conjugates for functions defined in X^* with respect to the dual system (X^*, X), then $f^{**} = (f^*)^* = \operatorname{cl}\operatorname{conv} f$ whenever $\operatorname{cl}\operatorname{conv} f$ (or equivalently f^*) is proper.

With respect to the dual system (X, X^*), the *polar of* $A \subset X$ is $A^\circ := \{x^* \in X^* \mid |\langle x, x^* \rangle| \leq 1,\ \forall x \in A\}$ while the *orthogonal of* A is $A^\perp := \{x^* \in X^* \mid \langle x, x^* \rangle = 0,\ \forall x \in A\}$; similarly, the polar of $B \subset X^*$ is $B^\circ := \{x \in X \mid |\langle x, x^* \rangle| \leq 1,\ \forall x^* \in B\}$ while the orthogonal of B is $B^\perp := \{x \in X \mid \langle x, x^* \rangle = 0,\ \forall x^* \in B\}$.

Let $Z := X \times X^*$. Consider the *coupling* function

$$c : Z \to \mathbb{R}, \quad c(z) := \langle x, x^* \rangle \text{ for } z := (x, x^*) \in Z.$$

It is known that the topological dual of $(Z, \tau \times w^*)$ can be (and will be) identified with Z by the coupling

$$z \cdot z' := \langle z, z' \rangle := \langle x, x'^* \rangle + \langle x', x^* \rangle \quad \text{for } z = (x, x^*),\ z' = (x', x'^*) \in Z.$$

With respect to the natural dual system (Z, Z) induced by the previous coupling, the Fitzpatrick function of $T : X \rightrightarrows X^*$ has the form

$$\varphi_T : Z \to \overline{\mathbb{R}}, \quad \varphi_T(z) = \sup\{z \cdot \alpha - c(\alpha) \mid \alpha \in \operatorname{Graph} T\},$$

while the conjugate of $f : Z \to \overline{\mathbb{R}}$ is denoted by

$$f^\square : Z \to \overline{\mathbb{R}}, \quad f^\square(z) = \sup\{z \cdot z' - f(z') \mid z' \in Z\},$$

and $f^{\square\square} = \operatorname{cl}_{\tau \times w^*} \operatorname{conv} f$ whenever $f^\square$ (or $\operatorname{cl}_{\tau \times w^*} \operatorname{conv} f$) is proper.

Note the identity

$$\inf_{\alpha \in \operatorname{Graph} T} c(z - \alpha) = (c - \varphi_T)(z). \tag{2}$$