J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Surjectivity of Euler operators on temperate distributions

Dietmar Vogt

Bergische Universität Wuppertal, Dept. of Math., Gauß-Str. 20, D-42119 Wuppertal, Germany

ARTICLE INFO

Article history:
Received 12 February 2018
Available online xxxx
Submitted by J. Bonet

Keywords:
Euler differential operators
Temperate distributions
Global solvability
C^∞-functions of exponential decay

ABSTRACT

Euler operators are partial differential operators of the form $P(\theta)$ where P is a polynomial and $\theta_j = x_j\partial/\partial x_j$. We show that every non-trivial Euler operator is surjective on the space of temperate distributions on R^d. This is in sharp contrast to the behaviour of such operators when acting on spaces of differentiable or analytic functions.

In the present note we study Euler differential operators on the space $\mathscr{S}'(\mathbb{R}^d)$ of temperate distributions on $\mathbb{R}^d$. These are operators of the form $P(\theta)$ where P is a polynomial and $\theta_j = x_j\,\partial/\partial x_j$. We show that every Euler operator is surjective on $\mathscr{S}'(\mathbb{R}^d)$ which is in sharp contrast to the behaviour in spaces of differentiable functions since the operator $P(\theta)$ is, in general, singular at the coordinate hyperplanes. Even for $d = 1$ the simple example of θ acting on $C^\infty(\mathbb{R})$ shows that surjectivity there is in general impossible. There are natural necessary conditions for a function to be in the range of an operator $P(\theta)$, solvability under these conditions has been shown in Domański–Langenbruch [2]. For real analytic functions the situation is even more complicated, see [1]. As an example our result implies the following: if g is a polynomial function on $\mathbb{R}^d$ then the equation $P(\theta)f = g$ may not have a C^∞-solution f on $\mathbb{R}^d$ but it will always have a temperate distribution f as solution on $\mathbb{R}^d$. We first study partial differential operators $P(\partial)$ with constant coefficients on the space $Y(\mathbb{R}^d)$ of C^∞-functions with exponential decay on $\mathbb{R}^d$ and on its dual the space $Y(\mathbb{R}^d)'$ the space of distributions with exponential growth. We show that every non-trivial operator $P(\partial)$ is surjective on $Y(\mathbb{R}^d)'$. By the exponential diffeomorphism this implies the surjectivity of $P(\theta)$ on the space $\mathscr{S}'(Q)$ of temperate distributions on the positive quadrant on $\mathbb{R}^d$, hence surjectivity on $\mathscr{S}'(\mathbb{R}^d)$ up to a distribution with support in the union of coordinate hyperplanes. By a method similar to the one used in [2] we then show the result by induction on the dimension.

E-mail address: dvogt@math.uni-wuppertal.de.

https://doi.org/10.1016/j.jmaa.2018.06.063

1. Preliminaries

We use the following notation $\partial_j = \partial/\partial x_j$, $\theta_j = x_j\,\partial_j$ and $D_j = -i\partial_j$. For a multiindex $\alpha \in \mathbb{N}_0^d$ we set $\partial^\alpha = \partial_1^{\alpha_1}..\partial_d^{\alpha_d}$, likewise for θ^α and D^α. For a polynomial $P(z) = \sum_\alpha c_\alpha z^\alpha$ we consider the *Euler operator* $P(\theta) = \sum_\alpha c_\alpha \theta^\alpha$ and also the operators $P(\partial)$ and $P(D)$, defined likewise.

$P(\theta)$ and $P(\partial)$ are connected in the following way. We set for $x \in \mathbb{R}^d$

$$\mathrm{Exp}(x) = (\exp(x_1), .., \exp(x_d)).$$

Exp is a diffeomorphism from $\mathbb{R}^d$ onto $Q := (0, +\infty)^d$. Therefore

$$C_{\mathrm{Exp}} : f \longrightarrow f \circ \mathrm{Exp}$$

is a linear topological isomorphism from $C^\infty(Q)$ onto $C^\infty(\mathbb{R}^d)$. For $f \in C^\infty(Q)$ we have $P(\partial)(f \circ \mathrm{Exp}) = (P(\theta)f) \circ \mathrm{Exp}$ that is $P(\partial) \circ C_{\mathrm{Exp}} = C_{\mathrm{Exp}} \circ P(\theta)$. In this way solvability properties of $P(\theta)$ on $C^\infty(Q)$ can be reduced to solvability properties of $P(\partial)$ on $C^\infty(\mathbb{R}^d)$. This has been done in [9]. We apply the same argument to the space $\mathscr{S}(Q)$ where $\mathscr{S}(Q) = \{f \in \mathscr{S}(\mathbb{R}^d) : \operatorname{supp} f \subset \overline{Q}\}$ and $\mathscr{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^d$.

Throughout the paper we use standard notation of Functional Analysis, in particular, of distribution theory, and of the theory of partial differential operators. For unexplained notation we refer to [3], [5], [6], [7], [8].

2. Distributions with exponential growth

We start with studying partial differential operators on $\mathbb{R}^d$ and we will transfer our results by the exponential diffeomorphism to results on Euler operators on Q. We set

$$\begin{aligned} Y(\mathbb{R}^d) &:= \{f \in C^\infty(\mathbb{R}^d) : \sup_x |f^{(\alpha)}(x)|\, e^{k|x|} < \infty \text{ for all } \alpha \text{ and } k \in \mathbb{N}\} \\ &= \{f \in C^\infty(\mathbb{R}^d) : \sup_x |f^{(\alpha)}(x)|\, e^{x\eta} < \infty \text{ for all } \alpha \text{ and } \eta \in \mathbb{R}^d\} \end{aligned}$$

with its natural topology. Here $x\eta = \sum_j x_j \eta_j$ and $|x| := |x|_1$.

Then $Y(\mathbb{R}^d)$ is a Fréchet space, closed under convolution and $P(\partial)$ is a continuous linear operator in $Y(\mathbb{R}^d)$ for every polynomial P. $\mathscr{D}(\mathbb{R}^d) \subset Y(\mathbb{R}^d)$ as a dense subspace, hence $Y(\mathbb{R}^d)' \subset \mathscr{D}'(\mathbb{R}^d)$. We obtain

Lemma 2.1. $C_{\mathrm{Exp}}(\mathscr{S}(Q)) = Y(\mathbb{R}^d)$.

Proof. We first claim that

$$(f \circ \mathrm{Exp})^{(\alpha)} = \sum_{\beta \le \alpha} a_\beta\, (f^{(\beta)} \circ \mathrm{Exp})\, \mathrm{Exp}^\beta$$

with $a_\alpha = 1$ and this is shown by induction.

This implies that for $f \in \mathscr{S}(Q)$ we have

$$\sup_{x \in \mathbb{R}^d} |(f \circ \mathrm{Exp})^{(\alpha)}(x)|\, e^{k|x|} \le \sum_{\beta \le \alpha} a_\beta \sup_{\xi \in Q} |f^{(\beta)}(\xi)||\xi|^{|\beta|+k} < +\infty$$

for all α and $k \in \mathbb{N}$.

Please cite this article in press as: D. Vogt, Surjectivity of Euler operators on temperate distributions, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.06.063