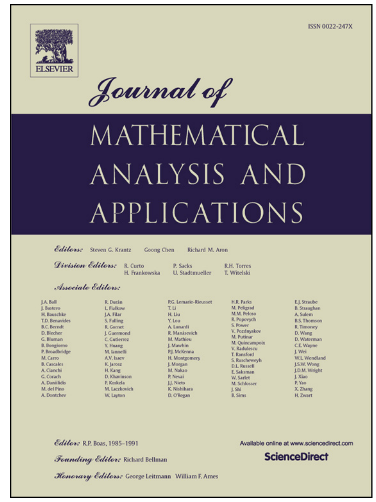

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Carlson's Theorem for Different Measures

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Abstract

We use an observation of Bohr connecting Dirichlet series in the right half plane $\mathbb{C}_+$ to power series on the polydisk to interpret Carlson's theorem about integrals in the mean as a special case of the ergodic theorem by considering any vertical line in the half plane as an ergodic flow on the polytorus. Of particular interest is the imaginary axis because Carlson's theorem for Lebesgue measure does not hold there. In this note, we construct measures for which Carlson's theorem does hold on the imaginary axis for functions in the Dirichlet series analog of the disk algebra $\mathcal{A}(\mathbb{C}_+)$.

Keywords: Dirichlet Series, Ergodic Theorem

1. Introduction

In 1913, Bohr [2] observed that one may connect Dirichlet series converging on the right half plane $\mathbb{C}_+ = \{s \in \mathbb{C} : \Re(s) > 0\}$ to power series on the infinite polydisk using the correspondence

$$z_1 = 2^{-s}, z_2 = 3^{-s}, \dots, z_j = p_j^{-s}, \dots$$

where p_j denotes the jth prime. For a Dirichlet series $f = \sum_{n=1}^{\infty} a_n n^{-s}$, we can use the fundamental theorem of arithmetic to factor each integer n uniquely and then represent f by a power series F in the variables $\{z_j\}$. As discussed in [5], the Bohr correspondence also allows us to consider any vertical line in $\mathbb{C}$ as an ergodic flow on the infinite-dimensional polytorus $\mathbb{T}^\infty$:

$$(e^{i\theta_1}, e^{i\theta_2}, \dots) \mapsto (p_1^{-it}e^{i\theta_1}, p_2^{-it}e^{i\theta_2}, \dots) \in \mathbb{T}^\infty$$

and in particular, the imaginary axis maps to the boundary of the infinite polydisk (of radius one.) We would like to compare a "space average" of the power series F on $\mathbb{T}^\infty$ to a "time average" of the Dirichlet series f on the ergodic flow described above. For this question, we consider $\mathcal{H}^\infty$, the Banach space of Dirichlet series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1.1}$$

that converge to bounded analytic functions on $\mathbb{C}_+$.

A theorem of Carlson [3] tells us about the limit in the mean of a Dirichlet series on an ergodic flow for $\sigma > 0$.

Theorem 1 (Carlson's Theorem)**.** *If a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges in the right half plane $\mathbb{C}_+$ and is bounded in every half plane $\Re(s) \geq \delta$ for $\delta > 0$, then for each $\sigma > 0$*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |f(\sigma + it)|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma}. \tag{1.2}$$

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