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Stability and superconvergence of MAC schemes for time dependent Stokes equations on nonuniform grids

Xiaoli Li, Hongxing Rui *

School of Mathematics, Shandong University, Jinan, Shandong 250100, China

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A B S T R A C T

In this paper, two MAC schemes are introduced and analyzed to solve the time dependent Stokes equations on nonuniform grids. One scheme is the Euler backward scheme with first order accuracy in time increment while the other one is the Crank Nicolson scheme with second order accuracy in time increment. By constructing an auxiliary function depending on the velocity and discretizing parameters, we obtain the second order superconvergence in L^2 norm for both velocity and pressure. Besides, second order superconvergence for some terms of H^1 norm of the velocity on nonuniform grids is obtained. Finally, some numerical experiments are presented to show that the convergence rates are in agreement with the theoretical analysis.

1. Introduction

The Stokes and Navier–Stokes equations are important basic models in flow dynamics and often employed in scientific computation. They have been widely applied in physics, chemistry and engineering. Usually, it is difficult to get analytical solutions of the Stokes and Navier–Stokes equations. Thus, numerical methods become a powerful tool for solving these equations.

There has been an enormous amount of research work devoted to the numerical solution of the Stokes and Navier–Stokes equations. Among the many numerical methods that are available for solving the Stokes and Navier–Stokes equations, one of the best and simplest is the marker and cell method (Girault and Lopez [3]). The same point view has been shared by Han and Wu [6]: "It is well known that the marker and cell (MAC) is one of the simplest and most effective numerical schemes for solving Stokes equations and Navier–Stokes equations." The MAC method was introduced by Lebedev [8] and Daly et al. [19] in 1960s, and has been widely used in engineering applications as evidenced by the fact that it is the basis of many flow packages [14]. The MAC scheme has the ability to enforce the incompressibility constraint

* Corresponding author.
E-mail addresses: xiaolisdu@163.com (X. Li), hxrui@sdu.edu.cn (H. Rui).

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of the velocity field point-wisely. Moreover, it has been shown to locally conserve the mass, momentum and kinetic energy [15,16]. The MAC method is a class of finite volume method on rectangular cells with pressure approximated at the cell center, the x-component of velocity approximated at the midpoint of vertical edges of the cell, and the y-component of velocity approximated at the midpoint of horizontal edges of the cell. The MAC method can also be interpreted as a mixed finite-element method coupled with a quadrature formula. This interpretation can be found in Girault and Raviart [4,5], where the mixed method is analyzed. And the error estimates for the MAC method was derived in [3].

In 1998, Han and Wu [6] showed that the MAC scheme can be obtained from a new mixed finite element method. Kanschat established that discontinuous Galerkin discretizations in their lowest order version were very similar to the MAC finite difference scheme and presented a way to interpolate solutions obtained by the MAC scheme to obtain a pointwise divergence-free function for the Stokes equations in [7]. Inspired by the work of Kanschat [7], Minev [12] demonstrated that some well-known finite-difference schemes can be interpreted within the framework of the local discontinuous Galerkin (LDG) methods using the low-order piecewise solenoidal discrete spaces.

All these papers [3,6,7,14] proved that the MAC method has first order convergence $O(h)$ for both the velocity (in H^1 norm) and the pressure (in L^2 norm) on uniform rectangular meshes. But the numerical results presented by Nicolaides [14] showed that the velocity is $O(h^2)$ without any proof.

Recently, Li and Sun [10] studied a MAC scheme for stationary Stokes equations on nonuniform grid and proved the stability of the velocity. To our knowledge, it is the first paper trying to analyze the second order convergence for the velocity. But in their paper the convergence analysis for the velocity is based on the assumption that the pressure has second order convergence which has not been proved. Rui and Li [17] found a discrete auxiliary velocity function depending on the exact velocity and discretizing parameters and proved that the approximate velocity of the MAC method converged to the auxiliary velocity with second order accuracy in discrete H^1 norms. By this, the second order superconvergence in the L^2 norm for both velocity and pressure was obtained. Besides, it was easily obtained the second order superconvergence for some terms of the H^1 norm of the velocity by using this method.

In this paper, two MAC schemes are introduced and analyzed to solve the time dependent Stokes equations on nonuniform grids. One scheme is the Euler backward scheme with first order accuracy in time increment while the other one is the Crank Nicolson scheme with second order accuracy in time increment. By establishing the LBB condition, the stability for both the velocity and pressure of the two MAC schemes is derived rigorously. Inspired by the analysis technique in [1,9,13], we also obtain the second order superconvergence in L^2 norm for both velocity and pressure by constructing an auxiliary function depending on the velocity and discretizing parameters. Besides, second order superconvergence for some terms of H^1 norm of the velocity on nonuniform grids is obtained. Our analysis is given for two dimensional problems. Because we analyze the error terms direction by direction, similarly results can be obtained for three dimensional problems.

The paper is organized as follows. In Section 2 we give the problem and some preliminaries. In Section 3 we present the MAC schemes and stability. In Section 4 we demonstrate error analysis for discrete schemes. In Section 5 some numerical experiments using the corresponding MAC schemes are carried out, which show that the convergence rates are in agreement with the theoretical analysis.

Throughout the paper we use C, with or without subscript, to denote a positive constant, which could have different values at different appearances.

2. The problem and some preliminaries

In this paper, we consider the time dependent Stokes equations with homogenerous boundary condition in two dimensional domain for an incompressible fluid.

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