Doctopic: Miscellaneous ARTICLE IN PRESS YJMAA:22385

J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Magnetic curves in tangent sphere bundles II

Jun-ichi Inoguchi [a], Marian Ioan Munteanu [b,*]

[a] Institute of Mathematics, University of Tsukuba, 1-1-1 Tennodai Tsukuba, Ibaraki, 305-8571, Japan
[b] University 'Al. I. Cuza' of Iasi, Faculty of Mathematics, Bd. Carol I, no. 11, 700506 Iasi, Romania

ARTICLE INFO

Article history:
Received 18 January 2018
Available online xxxx
Submitted by J. Xiao

Dedicated to professor Vasile Oproiu on the occasion of his 77th anniversary

Keywords:
Magnetic field
Tangent sphere bundle
α-Sasakian manifold
Periodic curve

ABSTRACT

We study contact magnetic curves in the unit tangent sphere bundle over the Euclidean plane. In particular, we obtain all contact magnetic curves which are slant.

1. Introduction and preliminaries

As is well known, unit tangent sphere bundle over Riemannian manifolds admits the so-called standard contact metric structure. In our previous paper [12] we have developed a general theory of magnetic curves in unit tangent sphere bundles. In addition we studied magnetic curves in the unit tangent bundle $U\mathbb{S}^2$ of the unit 2-sphere $\mathbb{S}^2$. As a continuation of [12], in this paper, we study magnetic curves in the unit tangent sphere bundle $U\mathbb{E}^2$ of the Euclidean plane $\mathbb{E}^2$. In particular, we obtain all contact normal magnetic curves on $U\mathbb{E}^2$, which satisfy the conservation law. Because the unit tangent sphere bundle $U\mathbb{E}^2$ may be identified as a contact metric manifold with the motion group $E(2)$ of the Euclidean plane $\mathbb{E}^2$, we do some investigations in $E(2)$.

1.1. Magnetic curves

Magnetic curves represent, in physics, the trajectories of charged particles moving on a Riemannian manifold under the action of the magnetic fields. Let (M, g) be a Riemannian manifold and let F be a

* Corresponding author.
E-mail addresses: inoguchi@math.tsukuba.ac.jp (J. Inoguchi), marian.ioan.munteanu@gmail.com (M.I. Munteanu).

https://doi.org/10.1016/j.jmaa.2018.06.069

closed 2-form on M (often called a *magnetic field* on M). A *magnetic curve* represents a solution of a second order differential equation

$$\nabla_{\gamma'}\gamma' = \phi\gamma', \tag{1.1}$$

where ∇ denotes the Levi-Civita connection on M and ϕ is a skew-symmetric $(1, 1)$ tensor field associated to F, that is $F(X, Y) = g(\phi X, Y)$ for any vector fields X, Y on M. See e.g. [1]. Such curves are sometimes called also *magnetic geodesics* since the Lorentz equation generalizes the equation of geodesics under arc-length parametrization, namely, $\nabla_{\gamma'}\gamma' = 0$. The equation (1.1) is usually known as the *Lorentz equation*. However, in contrast to the geodesics, magnetic curves cannot be rescaled, because the trajectory of a charged particle depends on the speed $|\gamma'|$. Nevertheless, magnetic curves have constant speed, and hence constant energy, since $\frac{d}{ds}\, g(\gamma', \gamma') = 2g(\phi\gamma', \gamma') = 0$.

And now, as usual, we restrict our investigation to a single energy level and we consider only unit speed magnetic curves together with a *strength* $q \in \mathbb{R}$. Therefore, from now on, we study *normal magnetic curve* (i.e. unit speed) satisfying the Lorentz equation

$$\nabla_{\dot\gamma}\dot\gamma = q\ \phi\dot\gamma, \tag{1.2}$$

where by `dot` we denote the derivative with respect to the arc-length parameter s.

1.2. Almost contact metric structures

A (φ, ξ, η)-*structure* on a manifold M is defined by a field φ of endomorphisms of tangent spaces, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

If (M, φ, ξ, η) admits a compatible Riemannian metric g, namely

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all X, $Y \in \Gamma(TM)$, then M is said to have an *almost contact metric structure*, and $(M, \varphi, \xi, \eta, g)$ is called an *almost contact metric manifold*. Consequently, we have that ξ is unitary and η is metrically dual to ξ, *i.e.*, $\eta(X) = g(\xi, X)$, for any $X \in \Gamma(TM)$. The vector field ξ is often called the Reeb vector field, even though, this name is used for the contact case.

We define a 2-form Ω on $(M, \varphi, \xi, \eta, g)$ by

$$\Omega(X, Y) = g(X, \varphi Y),$$

for all X, $Y \in \Gamma(TM)$, called the *fundamental 2-form* of the almost contact metric structure (φ, ξ, η, g).

The fundamental 2-form is not always closed. However, there are several classes of almost contact metric manifolds with closed fundamental 2-form. For more details see e.g. [3].

Let us remember some more definitions: An almost contact metric manifold is said to be:

(1) A *contact metric manifold* if $\Omega = d\eta$.
(2) An α-*Sasakian manifold* if there exists a constant α such that it satisfies

$$(\nabla_X\varphi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\}$$