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Sharp approximations for the complete elliptic integrals of the second kind by one-parameter means ☆

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A B S T R A C T

Let $E(r)$ be the complete elliptic integrals of the second kind and $J_p(a,b)$ be the one-parameter mean of two distinct positive numbers a and b defined by

$$J_p(a,b)=\frac{p}{p+1}\frac{a^{p+1}-b^{p+1}}{a^p-b^p}\text{ for }p\neq -1,0.$$

In this study using the recurrence method with the sign rule of a special power series, we prove that the double inequality

$$\frac{7\pi}{22}J_{2/(\pi-2)}(1,r')<\frac{2}{\pi}E(r)<J_{2/(\pi-2)}(1,r')$$

holds for $r'=\sqrt{1-r^2}\in(0,1)$, where the upper bound is sharp, while the lower bound is related to the Archimedes' approximation $22/7$ for π. Using the same method, we also easily show that both of the functions

$$r^2\mapsto\frac{(2/\pi)E(r)-J_{7/4}(1,r')}{r^{12}}\text{ and }r^2\mapsto\frac{1}{r^{10}}\left(1-\frac{1-(2/\pi)E(r)}{1-J_{7/4}(1,r')}\right)$$

are absolutely monotonic on $(0,1)$, which greatly improves the known results. Moreover, it should be emphasized that our method may be used to explore special functions related to hypergeometric functions.

1. Introduction

For $r\in(0,1)$, the well-known complete elliptic integrals of the first and second kinds [14], [15] are defined by

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$$K(r)=\int_0^{\pi/2}\frac{1}{\sqrt{1-r^2\sin^2 t}}dt,\ \ K\left(0^+\right)=\frac{\pi}{2},\ \ E\left(1^-\right)=\infty,$$

$$E(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2 t}dt,\ \ E\left(0^+\right)=\frac{\pi}{2},\ \ E\left(1^-\right)=1,$$

respectively, which are related to the Gaussian hypergeometric function

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\quad |z|<1,$$

where $a,b,c\in\mathbb{R}$ with $c\neq 0,-1,-2,...,$ $(a)_n$ is defined by $(a)_0=1$ for $a\neq 0$ and

$$(a)_n=a(a+1)\cdots(a+n-1)=\frac{\Gamma(n+a)}{\Gamma(a)},\quad a\neq 0.$$

More precisely, we have

$$K=K(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}r^{2n},$$

$$E=E(r)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(n!)^2}r^{2n},$$

which can be expressed as

$$K(r)=\frac{\pi}{2}\sum_{n=0}^{\infty}W_n^2r^{2n}\ \text{ and }\ E(r)=-\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{W_n^2}{2n-1}r^{2n},\tag{1.1}$$

where

$$W_n=\frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}\tag{1.2}$$

is the Wallis ratio.

It is well known that the complete elliptic integrals have many important applications in physics, engineering, geometric function theory, quasiconformal analysis, theory of mean values, number theory and other related fields [3], [4], [5], [7], [8], [9], [10], [11], [29], [32], [37].

Let $l(1,r)$ be the arc length of an ellipse with smilaxes 1 and $r\in(0,1)$. Then

$$l(1,r)=4E(r'),$$

where and in what follows $r'=\sqrt{1-r^2}$. Muir [27] presented a simple approximation for $l(1,r)$ by $2\pi A_{3/2}(1,r)$, where

$$A_p(a,b)=\left(\frac{a^p+b^p}{2}\right)^{1/p}\ \text{ if } p\neq 0 \text{ and } A_0(a,b)=\sqrt{ab}$$

is the classical power mean of positive numbers a and b. Tomkys [36] **provided** another power mean approximation $2\pi A_{q_0}(1,r)$, where $q_0=\ln 2/\ln(\pi/2)$, that is,