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Oscillation problems for Hill's equation with periodic damping

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ABSTRACT

This paper deals with the second-order linear differential equation $x'' + a(t)x' + b(t)x = 0$, where a and b are periodic coefficients. The main purpose is to present new criteria which guarantee that all nontrivial solutions are nonoscillatory and that those are oscillatory. Our nonoscillation theorem and oscillation theorem are proved by using the Riccati technique. In our theorem, the composite function of an indefinite integral of b and a suitable multiple-valued continuously differentiable function are focused, and the composite function of them plays an important role. The results obtained here include a result by Kwong and Wong [15] and a result by Sugie and Matsumura [26]. An application to a equation of Whittaker–Hill type is given to show the usefulness of our results. Finally, simulations are also attached to illustrate that our oscillation criterion is sharp.

1. Introduction

We consider the second-order linear differential equation

$$x'' + a(t)x' + b(t)x = 0, \tag{1.1}$$

where $a, b : [0, \infty) \to \mathbb{R}$ are continuous and periodic functions with period $T > 0$. Equation (1.1) has obviously the trivial solution $x \equiv 0$. Since all other solutions also exist in the future, they are divided into two groups as follows. A nontrivial solution x of (1.1) is said to be *oscillatory* if it has an infinite number of zeros on the interval $[0, \infty)$. Otherwise, the nontrivial solution is said to be *nonoscillatory*. In other words, if x is a nonoscillatory solution of (1.1), then there exists a $t^* \geq 0$ such that $x(t) > 0$ for $t \geq t^*$ or $x(t) < 0$ for $t \geq t^*$.

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We can consider equation (1.1) to be a motion equation having a very simple form. For this reason, equation (1.1) has been widely studied as models which appear not only in pure mathematics but also in various fields. We can easily find the literatures related to oscillation theory for equation (1.1) and more generalized equations including (1.1) (refer to [1–3,6,7,18,22,23,27,28,30–34]).

In the special case when a is equal to zero identically, equation (1.1) becomes the differential equation

$$x'' + b(t)x = 0 \tag{1.2}$$

which is famous in astrophysics. This equation was proposed by George William Hill to analyze the orbit of the moon originally (see [9]). Equation (1.2) is called Hill's equation named after him.

Let p be any continuous and periodic function with period $T > 0$. By taking into account the definite integral of p from 0 to T, we can define the following family of functions. The periodic function p is said to be periodic of *mean value zero* if p is not identically zero and

$$\int_0^T p(t)dt = 0.$$

Let us denote by $\mathscr{F}_{[\mathrm{MVZ}]}$ the family of functions which are periodic of mean value zero. About other applications of Hill's equation, refer to [19,20]. We can find various results about the oscillation problem of (1.2) in many literatures (for example, see [3,15–17,27]).

It is well known that if b belongs to $\mathscr{F}_{[\mathrm{MVZ}]}$, then all nontrivial solutions of (1.2) are oscillatory (for the proof, see [3, p. 25]). For example, if $b(t) = \sin t$ (or $b(t) = \cos t$), then all nontrivial solutions of (1.2) are oscillatory. However, even if a and b belong to $\mathscr{F}_{[\mathrm{MVZ}]}$, all solutions of (1.1) are not always oscillatory. By focusing on this fact, Kwong and Wong [15] have studied oscillation and nonoscillation of equation (1.1). Let B be an indefinite integral of b. Then, their nonoscillation criterion [15, Theorem 1] can be stated as follows.

Theorem A. *Suppose that b belongs to $\mathscr{F}_{[\mathrm{MVZ}]}$. If*

$$(B(t) - a(t))B(t) \le 0 \quad \textit{for} \ \ 0 \le t \le T,$$

then all nontrivial solutions of (1.1) *are nonoscillatory.*

Remark 1.1. It is clear that the difference of two indefinite integrals of b is constant. The periodic coefficient b belongs to $\mathscr{F}_{[\mathrm{MVZ}]}$ if and only if all indefinite integrals of b are periodic. The condition that b belongs to $\mathscr{F}_{[\mathrm{MVZ}]}$ is used only to show that B is a periodic function with period T. Hence, we can rewrite the statement of Theorem A as "If B is periodic and

$$(B(t) - a(t))B(t) \le 0 \quad \text{for} \ \ 0 \le t \le T,$$

then all nontrivial solutions of (1.1) are nonoscillatory."

A typical example that can be applied to Theorem A is

$$x'' + (\sin t)x' + (\cos t)x = 0. \tag{1.3}$$

Since $b(t) = \cos t$ in equation (1.3), we can choose $B(t) = \sin t$. Hence, we have

$$(B(t) - a(t))B(t) = (\sin t - \sin t)\sin t = 0 \quad \text{for} \ \ 0 \le t \le 2\pi.$$

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