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Analytical approximations of non-linear SDEs of McKean–Vlasov type

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A B S T R A C T

We provide analytical approximations for the law of the solutions to a certain class of scalar McKean–Vlasov stochastic differential equations (MKV-SDEs) with random initial datum. "Propagation of chaos" results ([15]) connect this class of SDEs with the macroscopic limiting behavior of a particle, evolving within a mean-field interaction particle system, as the total number of particles tends to infinity. Here we assume the mean-field interaction only acting on the drift of each particle, this giving rise to a MKV-SDE where the drift coefficient depends on the law of the unknown solution. By perturbing the non-linear forward Kolmogorov equation associated to the MKV-SDE, we perform a two-steps approximating procedure that decouples the McKean–Vlasov interaction from the standard dependence on the state-variables. The first step yields an expansion for the marginal distribution at a given time, whereas the second yields an expansion for the transition density. Both the approximating series turn out to be asymptotically convergent in the limit of short times and small noise, the convergence order for the latter expansion being higher than for the former. Concise numerical tests are presented to illustrate the accuracy of the resulting approximation formulas. The latter are expressed in semi-closed form and can be then regarded as a viable alternative to the numerical simulation of the large-particle system, which can be computationally very expensive. Moreover, these results pave the way for further extensions of this approach to more general dynamics and to high-dimensional settings.

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1. Introduction

Model Consider the non-linear diffusion

$$\begin{cases} \mathrm{d}X_t = \mathbb{E}[b(x, X_t)]|_{x=X_t}\mathrm{d}t + \sigma \mathrm{d}W_t, & t > 0 \\ X_0 = Y. \end{cases} \tag{1.1}$$

Here, W is a scalar Brownian motion and Y is a square integrable random variable, independent of W. Throughout the paper, we assume that there exist two positive constants $M, \bar{\sigma} > 0$ such that the following standing assumptions hold:

[Hyp-b.0] $b : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is a globally Lipschitz function, and is bounded by $M\sigma^2$;
[Hyp-σ] The diffusion coefficient σ is such that $0 < \sigma \leq \bar{\sigma}$.

For high order expansions, **[Hyp-**b.0] will be reinforced by adding the following further assumption, for a given $N \in \mathbb{N}$, $N \geq 1$.

[Hyp-b.N] For any $y \in \mathbb{R}$, the function $b(\cdot, y) \in C^N(\mathbb{R})$ with all the derivatives $\partial_1^n b(\cdot, \cdot)$ up to order N being measurable and bounded by $M\sigma^2$. Moreover, $\partial_1 b(\cdot, \cdot)$ is continuous.

Such non-linear SDEs, where the coefficient b of the equation depends not only on the state of the solution at time t, but also on its whole distribution, is a particular case of a class of SDEs known as *McKean-type non-linear diffusions*. It is well known that, under **[Hyp-**b.0], Eq. (1.1) admits a unique strong solution (see for instance [15]). The extra assumption **[Hyp-**σ], along with other additional regularity and boundedness assumptions on b, will be used to derive expansions for the density of the distribution of X_t. In particular, the need for the constant $\bar{\sigma}$ will be clarified in the sequel. Loosely speaking, it will allow to prove sharp error estimates not only for small times, but also for small σ.

Background results and main contributions So far, the study of numerical approximations of SDEs of McKean-type has been mainly conducted under the point of view of time discretization and simulation through an interacting particles system. References are numerous and we refer to [13,5,2,16,17] among others. Recently, an alternative method using cubature formula has been investigated in [8]. Our approach is quite different and relies on analytical expansions; to the best of our knowledge this is fully novel in this context. We emphasize that during the last decade, there has been an increasing gain of interest in the study of SDEs of McKean-type, with new applications ranging from modeling economic interactions and mean-field games [7,6], to financial portfolio [4,11] and neuroscience [9]. The first main contribution of the paper is a semi-closed N-th order approximation $\bar{\bar{P}}_{N,t}$ for the density P_t of X_t, for which we are able to prove an asymptotic error bound (Theorem 2.9) that can be roughly summarized as

$$\left\| P_t - \bar{\bar{P}}_{N,t} \right\|_{L^1(\mathbb{R})} = O\left(\sigma^2 t\right)^{\frac{N+1}{2}} \qquad \text{as } \sigma^2 t \to 0^+.$$

The second main contribution is a family of semi-closed N-th order approximations $\bar{p}_N^{\bar{x}}(s, \xi; t, x)$ for the transition density $p(s, \xi; t, x)$ of X (seen as a time-inhomogeneous standard SDE), the latter depending on the previous approximation $\bar{\bar{P}}_{N,t}$ in a way that will be specified in Section 2. In this case we are able to prove an asymptotic result (Theorem 2.16) that roughly reads as

$$\left| \left(p - \bar{p}_N^{\bar{x}}\right)(s, \xi; t, x) \right| = e^{-\frac{(x-\xi)^2}{4\sigma^2(t-s)}} O\left(\sigma^2 t\right)^{\frac{N+1}{2}} \qquad \text{as } \sigma^2 t \to 0^+,$$