J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Boundedness of projected composition operators over the unit disc

Chong Zhao [1]

School of Mathematics, Shandong University, Jinan, Shandong 250100, People's Republic of China

ARTICLE INFO

Article history:
Received 14 April 2018
Available online xxxx
Submitted by J.A. Ball

Keywords:
Composition operator
Bounded operator
Hardy space
Bergman space

ABSTRACT

In the present paper, we carry forward R. Rochberg's work [4] on the boundedness of composition operators associated to continuous symbols of norm greater than 1. More general sufficient conditions and necessary conditions are given for the boundedness of $K_\varphi := P_{H^2} D_\varphi$ where $D_\varphi : \mathbb{C}[z] \to L^2(\mathbb{T})$ is the composition operator of the symbol φ.

1. Introduction

Let $\Omega \subset \mathbb{C}$ be a bounded domain, and μ be a measure supported on Ω that $\partial\Omega \subset \operatorname{supp} \mu$. Define on $\mathbb{C}[z]$ the inner product

$$\langle f, g \rangle_\mu = \int\limits_\Omega f(z)\overline{g(z)} \mathrm{d}\mu(z),$$

and denote by H_μ the completion of $\mathbb{C}[z]$ with respect to $\langle \cdot, \cdot \rangle_\mu$. It is well-known that H_μ can be seen as a subspace of $L^2(\mu)$. Suppose φ is a continuous function on supp μ, then $f \circ \varphi \in L^2(\mu)$ for each $f \in \mathbb{C}[z]$. Let P_{H_μ} denote the orthogonal projection from $L^2(\mu)$ onto H_μ. If there is a constant C making $||P_{H_\mu}(f \circ \varphi)||_\mu \le C||f||_\mu, \forall f \in \mathbb{C}[z]$, then the mapping $f \to P_{H_\mu}(f \circ \varphi)$ extends to a bounded operator $K_{\varphi,\mu} : H_\mu \to H_\mu$. We call φ the symbol of K_φ.

In the case H_μ be the Hardy space $H^2(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic, the boundedness of the composition operator $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D}), f \mapsto f \circ \varphi$ was extensive studied in [2]. In the literature [3], investigation was made into the case where $\varphi \in C(\mathbb{T})$ is not assumed analytic. They proved the compactness of

E-mail address: chong.zhao@sdu.edu.cn.

[1] Partially supported by NSFC (No. 11501329) and Shandong Province Natural Science Foundation ZR2014AQ009.

https://doi.org/10.1016/j.jmaa.2018.07.021
0022-247X/© 2018 Published by Elsevier Inc.

$$K_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D}), f \mapsto P_{H^2}(f \circ \varphi)$$

when $(1-|\varphi|^2)^{-1}$ is integrable, and carefully studied the boundedness of K_φ for the symbols $\varphi(z) = az + b\bar{z}, |a| + |b| \le 1$. In Rochberg's work [4], the symbols φ with supremum norm greater than 1 were considered. When $||\varphi||_\infty > 1$, $f \circ \varphi$ can not be defined in the obvious manner on $H^2(\mathbb{D})$, and usually K_φ is not expected to be bounded. Therefore it is interesting to find out which φ make K_φ bounded. Introducing the d-bar methods and Large Deviations, Rochberg proved some general sufficient conditions for the boundedness of K_φ, and gave some examples in which K_φ is not bounded. In general, K_φ is bounded provided the analytic part of φ is small in some sense.

In the present paper we reinterpret Rochberg's observations, and obtain some further results. To illustrate our results, take the $H^2(\mathbb{D})$ case as an example. Denote by dm the normalized Haar measure on $\mathbb{T} = \partial\mathbb{D}$. Let $\varphi(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$ be a continuous function on $\mathbb{T}$. For $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}$ denote by $s_n(z) = \sum_{k=-n}^{n} c_k z^k$ the partial sums and $\sigma_n(z) = \frac{1}{2n+1}\sum_{k=0}^{n} s_k(z)$ the Cesàro means of the Fourier series of φ. It is well known that $\{\sigma_n|_\mathbb{T}\}$ converges to φ uniformly. Suppose the sequence $\{\sigma_n(z)\}$ of functions converges uniformly to a function φ^* on some ring

$$R_\varphi := \{z \in \mathbb{C} : |z| \in [1, \rho_1)\}, 1 < \rho_1 \le \infty,$$

then φ^* is analytic on $R_\varphi - \mathbb{T}$. Analytically continue φ^* from $R_\varphi - \mathbb{T}$ to a maximal domain $\Omega'_{\varphi^*} \subset \mathbb{C}\backslash\overline{\mathbb{D}}$, then φ^* is defined and continuous on $\Omega_{\varphi^*} := \Omega'_{\varphi^*} \cup \mathbb{T}$. By the terminology simple closed curve we mean a continuous injective function on $\mathbb{T}$. If a curve γ has nonzero first derivative and k-th continuous derivative, we say it is a regular C^k-curve. Denote by $\mathfrak{P}_\varphi$ the set of all the rectifiable simple closed curves that is homotopic to $\mathrm{Id}_\mathbb{T}$ within Ω_{φ^*}, and $\mathfrak{P}^k_\varphi$ the subset of regular C^k-curves from $\mathfrak{P}_\varphi$.

We prove the following sufficient condition for the boundedness of K_φ.

Theorem 1.1. *Suppose there is a $\gamma \in \mathfrak{P}_\varphi$ such that $||\varphi^* \circ \gamma||_\infty < 1$, then K_φ extends to a Hilbert–Schmidt operator on $H^2(\mathbb{D})$.*

Then in Section 3 we give two general conditions under which K_φ cannot be extended to a bounded operator, which are almost converses to Theorem 1.1. Applying our results, we show in Examples 2.3 and 3.7 that for the symbol $\varphi(e^{i\theta}) = \frac{c}{(e^{i\theta}-2/3)(e^{i\theta}+3)}, \theta \in [0, 2\pi)$, K_φ is bounded if and only if $|c| < \frac{121}{36}$, which cannot be obtained from previous work.

The present paper is organized as follows. In Section 2 we focus on $H^2(\mathbb{D})$, and obtain sufficient conditions for the boundedness of K_φ. In Section 3, we give some general necessary conditions. Then we generalize our approach to other function spaces. In Section 4 we look at the Bergman space $L^2(\mathbb{D})$.

2. Sufficient conditions for boundedness of K_φ on $H^2(\mathbb{D})$

Let φ be a continuous function on $\mathbb{T}$. As in the introduction we suppose there is a maximal domain $\Omega'_{\varphi^*} \subset \mathbb{C}\backslash\overline{\mathbb{D}}$ with $\mathbb{T} \subset \partial\Omega'_{\varphi^*}$ and a continuous function φ^* defined on $\Omega_{\varphi^*} = \Omega'_{\varphi^*} \cup \mathbb{T}$, such that φ^* is analytic on Ω'_{φ^*} and coincides with φ on $\mathbb{T}$.

For integers $m, n \ge 0$, we have

$$\langle \varphi^m, z^n \rangle_\mathbb{T} = \frac{1}{2\pi}\int_0^{2\pi} \varphi^m(e^{i\theta})e^{-in\theta}\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\varphi^{*m}(e^{i\theta})}{e^{in\theta}}\frac{\mathrm{d}e^{i\theta}}{ie^{i\theta}}$$