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Yet another induction scheme for non-uniformly expanding transformations

Pedro L. Capett-Figueras, Fernando J. Sánchez-Salas *

Departamento de Matemáticas, Facultad Experimental de Ciencias, Universidad del Zulia, Avenida Universidad, Edificio Grano de Oro, Maracaibo, Venezuela

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ABSTRACT

We introduce a new induction scheme for non-uniformly expanding maps f of compact Riemannian manifolds, relying upon ideas of [33] and [10]. We use this induction approach to prove that the existence of a Gibbs–Markov–Young structure is a necessary condition for f to preserve an absolutely continuous probability with all its Lyapunov exponents positive.

Contents

* Corresponding author.
E-mail addresses: pcapett@demat-fecluz.org (P.L. Capett-Figueras), fjss@fec.luz.edu.ve (F.J. Sánchez-Salas).

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1. Introduction

A Borel probability μ is f-invariant if $\mu(f^{-1}(B)) = \mu(B)$ for every Borel subset $B \subset M$. Invariant probabilities provide quantitative information about the spatial distribution of orbits of f and are crucial in the study of statistical properties of the dynamics. See [19] and [16, Chapter 4].

Absolutely continuous invariant measures (**acim**) are of interest because they are generated by sets of positive Lebesgue measure, that is, *they are experimentally observable*.

This paper is concerned with *induction*, a useful notion aimed at the study of ergodic properties of dynamical systems. Namely, suppose that every point in a non-empty subset $B \subset M$ returns infinitely many often to B. This is the case, for example, of a set B with $\mu(B) > 0$, by Poincaré's recurrence theorem. See [19]. Then it is natural to consider the *induced dynamical system* $f^R : B \to B$ defined as a *first return map* $f^R(x) = f^{R(x)}(x)$, where $R(x) = \min\{n > 0 : f^n(x) \in B\}$ is the *first return time* from $x \in B$ to B.

First return maps were originally introduced by Poincaré to study nearby orbits of limit cycles. It was later, in the work of Kakutani [15], that they revealed their usefulness in the study of ergodic properties of the dynamics. Transformations induced by more general return times were also considered in the work of J. Neveu and Schweiger between 1969 and 1981. See references in [26].

To the best of our knowledge, Jakobson's theorem [13] was the first to show the close connection between induced maps and the existence of **acim**. Jakobson's theorem says that given a one-parameter family of real quadratic maps $f_a(x) = 1 - ax^2$, there exists a Cantor set Λ with positive Lebesgue measure such that for every $a \in \Lambda$, $f = f_a$ has an ergodic **acim** with positive Lyapunov exponent. See [13, Theorem A, Theorem B].

The idea of the proof is to construct Λ in such way that for every $a \in \Lambda$, there exists a closed interval J and an induced expanding Markov transformation $f^R : \bigcup_i J_i \to J$ an bounded distortion (see subsection 1.1) with integrable return-time R. By a folklore theorem f^R has a (unique) ergodic f^R-invariant **acim**. Integrability of return time allows to "coinduce" an f-invariant **acim** μ, by an standard procedure (see below). We refer to Yoccoz's College de France manuscript [35] for an up-to-date and quite clear exposition of this celebrated result.

Jakobson's paper inspired a number of results concerning the application of induction method to prove the existence of **acim** for different classes of one-dimensional dynamical systems. See for instance [9, Chapter V, Section 3], [12], [14], [20] and [34].

The next breakthrough in the induction approach were Lai Sang Young's *"horseshoes with infinitely many branches and variable return times"*. See [36], [37]. Generally speaking, Young's horseshoes are Cantor sets endowed with an induced return map with good hyperbolic and distortion properties with respect to a reference measure m. The return structures are made of infinitely many 'vertical strips' mapped onto 'horizontal strips' similarly to Smale's horseshoes. Projecting along stable invariant manifolds one gets a uniformly expanding map with good distortion properties $f^R : \bigcup_i \Delta_i \to \Delta$ defined over a decomposition of a Cantor set Δ with positive reference measure. These Markov structures are used to prove the existence of equilibrium measures, showing the connection between the speed of convergence to the equilibrium and its rates of mixing with the rate of decaying of return time tails.

The following is a partial sample of the growing body of literature on the applications of also called *Markov towers* or *Gibbs–Markov–Young structures* (GMY) in the study of statistical properties of the dynamics. See [1], [2], [4], [5], [6], [8], [10], [11], [22], [21], [23], [24], [25], [26], [28], [29].

In [36] Young describes informally diffeomorphisms exhibiting GMY structures as those being *"hyperbolic in large part of the phase space without being uniformly hyperbolic"*, however, up to the best of our knowledge, a precise characterization of the systems displaying Markov structures is still lacking.

Recently, it was conjectured that a $C^{1+\alpha}$ *diffeomorphism of compact Riemannian manifold has a GMY structure if and only if it is non-uniformly hyperbolic*, meaning a system with an invariant measure with non zero Lyapunov exponents. The conjecture was proved in [1] for a large class of non-uniformly expand-