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Doubly nonlocal system with Hardy–Littlewood–Sobolev critical nonlinearity

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ABSTRACT

This article concerns about the existence and multiplicity of weak solutions for the following nonlinear doubly nonlocal problem with critical nonlinearity in the sense of Hardy–Littlewood–Sobolev inequality

$$\begin{cases} (-\Delta)^s u = \lambda |u|^{q-2}u + \left(\displaystyle\int_\Omega \frac{|v(y)|^{2^*_\mu}}{|x-y|^\mu}\, dy \right) |u|^{2^*_\mu - 2}u \text{ in } \Omega \\ (-\Delta)^s v = \delta |v|^{q-2}v + \left(\displaystyle\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\, dy \right) |v|^{2^*_\mu - 2}v \text{ in } \Omega \\ \qquad u = v = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n > 2s$, $s \in (0,1)$, $(-\Delta)^s$ is the well known fractional Laplacian, $\mu \in (0,n)$, $2^*_\mu = \dfrac{2n-\mu}{n-2s}$ is the upper critical exponent in the Hardy–Littlewood–Sobolev inequality, $1 < q < 2$ and $\lambda, \delta > 0$ are real parameters. We study the fibering maps corresponding to the functional associated with $(P_{\lambda,\delta})$ and show that minimization over suitable subsets of Nehari manifold renders the existence of at least two non trivial solutions of $(P_{\lambda,\delta})$ for suitable range of λ and δ.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ (at least C^2), $n > 2s$ and $s \in (0,1)$. We consider the following nonlinear doubly nonlocal system with critical nonlinearity:

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$$
(P_{\lambda,\delta}) \begin{cases} (-\Delta)^s u = \lambda |u|^{q-2}u + \left(\displaystyle\int_\Omega \frac{|v(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}y \right) |u|^{2^*_\mu - 2} u \text{ in } \Omega \\ (-\Delta)^s v = \delta |v|^{q-2}v + \left(\displaystyle\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}y \right) |v|^{2^*_\mu - 2} v \text{ in } \Omega \\ \qquad u = v = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases}
$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n > 2s$, $s \in (0,1)$, $\mu \in (0,n)$, $2^*_\mu = \dfrac{2n-\mu}{n-2s}$ is the upper critical exponent in the Hardy–Littlewood–Sobolev inequality, $1 < q < 2$, $\lambda, \delta > 0$ are real parameters and $(-\Delta)^s$ is the fractional Laplace operator defined as

$$
(-\Delta)^s u(x) = 2C^n_s \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, \mathrm{d}y
$$

where P.V. denotes the Cauchy principal value and $C^n_s = \pi^{-\frac{n}{2}} 2^{2s-1} s \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)}$, Γ being the Gamma function. The fractional Laplacian is the infinitesimal generator of Lévy stable diffusion process and arise in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance, see [3] for instance.

In the local case, authors in [5] studied the existence of ground states for the nonlinear Choquard equation

$$
-\Delta u + V(x)u = \left(\int_\Omega \frac{|u(y)|^p}{|x-y|^\mu} \, \mathrm{d}y \right) |u|^{p-2} u \ \text{ in } \mathbb{R}^n, \tag{1.1}
$$

where $p > 1$ and $n \geq 3$. Recently, Ghimenti, Moroz and Schaftingen [16] proved the existence of least action nodal solution for the problem

$$
-\Delta u + u = (I_\alpha * |u|^2)u \text{ in } \mathbb{R}^n,
$$

where $*$ denotes the convolution and I_α denotes the Riesz potential. Further results related to Choquard equations can be found in the survey paper [24] and the references therein. Alves, Figueiredo and Yang [1] proved existence of a nontrivial solution via penalization method for the following Choquard equation

$$
-\Delta u + V(x)u = (|x|^{-\mu} * F(u))f(u) \text{ in } \mathbb{R}^n,
$$

where $0 < \mu < N$, $N = 3$, V is a continuous real valued function and F is the primitive of function f. In the nonlocal case, Choquard equations involving fractional Laplacian is a recent topic of research. Authors in [9] obtained regularity, existence, nonexistence, symmetry as well as decays properties for the problem

$$
(-\Delta)^s u + \omega u = (|x|^{\alpha-n} * |u|^p)|u|^{p-2}u \text{ in } \mathbb{R}^n,
$$

where $\omega > 0$, $p > 1$ and $s \in (0,1)$. Fractional Choquard equations also known as nonlinear fractional Schrödinger equations with Hartree-type nonlinearity arise in the study of mean field limit of weakly interacting molecules, physics of multi particle systems and the quantum mechanical theory, etc. These are recently studied by some authors in [6,10,22]. We also refer [2] and [4] for the qualitative behaviour of Choquard type problems in the semiclassical limit.

Concerning the boundary value problems involving the Choquard nonlinearity, the Brezis–Nirenberg type problem that is