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Zeros of combinations of the Riemann Ξ-function and the confluent hypergeometric function on bounded vertical shifts

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ABSTRACT

In 1914, Hardy proved that infinitely many non-trivial zeros of the Riemann zeta function lie on the critical line using the transformation formula of the Jacobi theta function. Recently the first author obtained an integral representation involving the Riemann Ξ-function and the confluent hypergeometric function linked to the general theta transformation. Using this result, we show that a series consisting of bounded vertical shifts of a product of the Riemann Ξ-function and the real part of a confluent hypergeometric function has infinitely many zeros on the critical line, thereby generalizing a previous result due to the first and the last authors along with Roy and Robles. The latter itself is a generalization of Hardy's theorem.

1. Introduction

Ever since the appearance of Riemann's seminal paper [23] in 1859, the zeros of the Riemann zeta function $\zeta(s)$ have been a constant source of inspiration and motivation for mathematicians to produce beautiful mathematics. While the Riemann Hypothesis, which states that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = 1/2$, has defied all attempts towards its proof as of yet, the beautiful area of analytic number theory has blossomed to what it is today because of these attempts.

One of the major breakthroughs in this area occurred with G.H. Hardy [14] proving that infinitely many non-trivial zeros of $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = 1/2$. Let $N_0(T)$ denote the number of non-trivial zeros lying on the critical line such that their positive imaginary part is less than or equal to T. Hardy and Littlewood [15] showed that $N_0(T) > AT$, where A is some positive constant. Selberg [26] remarkably improved this to $N_0(T) > AT \log T$. Levinson [17] further improved this by proving that more than one-third

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of the non-trivial zeros of $\zeta(s)$ lie on the critical line. Conrey [4] raised this proportion and showed at least 40.77% of the zeros lie on the critical line. This proportion was later improved by Bui, Conrey and Young [3], Feng [12], Robles, Roy and one of the authors [24]. Pratt and Robles [19] further improved the proportion to 41.49%. The current record is due to Pratt, Robles, Zaharescu and Zeindler [20] who showed that 41.729% of the zeros lie on the critical line.

The proof of Hardy's result in [14], which acted as a stimulus to the aforementioned developments, is well-known for its beauty and elegance. One of the crucial ingredients in his proof is the transformation formula satisfied by the Jacobi theta function. The latter is defined by

$$\vartheta(\lambda;\tau) := \sum_{n=-\infty}^{\infty} q^{n^2}\lambda^n,$$

where $q = \exp(\pi i\tau)$ for $\tau \in \mathbb{H}$ (upper half-plane) and $\lambda = e^{2\pi iu}$ for $u \in \mathbb{C}$. If we let $\tau = ix$ for $x > 0$ and $u = 0$, then the theta function becomes

$$\vartheta(1;ix) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x} =: 2\psi(x) + 1,$$

so that $\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$. The aforementioned transformation formula employed by Hardy in his proof is due to Jacobi and is given by [28, p. 22, Equation 2.6.3]

$$\sqrt{x}(2\psi(x)+1) = 2\psi\left(\frac{1}{x}\right) + 1, \tag{1.1}$$

which can be alternatively written as

$$\sqrt{a}\left(\frac{1}{2a} - \sum_{n=1}^{\infty} e^{-\pi a^2n^2}\right) = \sqrt{b}\left(\frac{1}{2b} - \sum_{n=1}^{\infty} e^{-\pi b^2n^2}\right) \tag{1.2}$$

for $\mathrm{Re}(a^2) > 0$, $\mathrm{Re}(b^2) > 0$ and $ab = 1$. The former is the same version of the theta transformation using which Riemann [23] derived the functional equation of the Riemann zeta function $\zeta(s)$ in the form [28, p. 22, eqn. (2.6.4)]

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

In fact, the functional equation of $\zeta(s)$ is known to be equivalent to the theta transformation. Another crucial step in Hardy's proof is the identity

$$\frac{2}{\pi}\int_0^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}}\cosh(\alpha t)\mathrm{d}t = e^{-\frac{1}{2}i\alpha} - 2e^{\frac{1}{2}i\alpha}\psi\left(e^{2i\alpha}\right), \tag{1.3}$$

where $\Xi(t)$ is the Riemann Ξ-function defined by

$$\Xi(t) = \xi\left(\tfrac{1}{2} + it\right)$$

with $\xi(s)$ being the Riemann ξ-function

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$