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High multiplicity of positive solutions for superlinear indefinite problems with homogeneous Neumann boundary conditions

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A R T I C L E I N F O

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Truth for Giulio Regeni (1988–2016), a young researcher of my homeland

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A B S T R A C T

We prove that a class of superlinear indefinite problems with homogeneous Neumann boundary conditions admits an arbitrarily high number of positive solutions, provided that the parameters of the problem are adequately chosen. The sign-changing weight in front of the nonlinearity is taken to be piecewise constant, which allows us to perform a sharp phase-plane analysis, firstly to study the sets of points reached at the end of the regions where the weight is negative, and then to connect such sets through the flow in the positive part. Moreover, we study how the number of solutions depends on the amplitude of the region in which the weight is positive, using the latter as the main bifurcation parameter and constructing the corresponding global bifurcation diagrams.

1. Introduction

We investigate existence and multiplicity of positive solutions for the following Neumann boundary value problem

$$\begin{cases} -u'' = \lambda u + a(t)u^p & \text{for } t \in (0,1), \\ u'(0) = 0 = u'(1), \end{cases} \tag{1.1}$$

where $p > 1$ and $\lambda < 0$ are constants and the weight $a(t)$ is a piecewise constant function of type

$$a(t) := \begin{cases} -c & \text{if } t \in (0,\alpha) \cup (1-\alpha,1), \\ b & \text{if } t \in [\alpha, 1-\alpha] \end{cases} \tag{1.2}$$

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with $\alpha \in (0, 1/2)$, $b > 0$ and $c > 0$. We will look for strong solutions of (1.1) in the sense that they are of class $\mathcal{C}^1([0,1]) \cap \mathcal{C}^2([0,\alpha)) \cap \mathcal{C}^2((\alpha, 1-\alpha)) \cap \mathcal{C}^2((1-\alpha, 1])$, which is the highest possible regularity, and, without further mention, we will consider only positive solutions throughout the whole paper.

Problems like (1.1)–(1.2), where the nonlinearity is of superlinear type, since $p > 1$, and the weight function changes sign, are known in the literature as *superlinear indefinite*. Such kind of problems, with Neumann (and even more general) boundary conditions have been widely treated in the last decades starting from [3,4,2] (see also the references therein and [1,13,16] for the case of Dirichlet boundary conditions). In these works, by using a variety of mathematical techniques that go from variational methods up to bifurcation and continuation theory, necessary and sufficient conditions for existence and some (low) multiplicity results have been obtained.

Recently, in [19], a new insight has been given to positive solutions of superlinear indefinite problems like the ones we study here. There, with the difference that Dirichlet inhomogeneous boundary conditions $u(0) = u(1) = M \in (0, +\infty]$ are considered (when $M = +\infty$ the boundary condition has to be understood in the limiting sense and the solutions are referred to as *large* or *blow-up solutions*), it has been proved that the structure of positive solutions can be extremely rich. Indeed, by using a topological shooting technique which, to the best of our knowledge, goes back to [20], it has been shown that, when λ is sufficiently negative, there exists a specific value of b, for which the problem possesses an arbitrarily high number of positive solutions. Moreover, by using b as the main bifurcation parameter (an idea which goes back to [16]), the structure of the global bifurcation diagrams has been determined.

Another situation was also known to produce high multiplicity of positive solutions for superlinear indefinite problems: precisely, when the weight function $a(t)$ in (1.1) has $n \in \mathbb{N}^*$ components where it is positive, separated by regions where it is negative. Indeed, according to some numerical observations, it was conjectured in [13] that, for homogeneous Dirichlet boundary conditions and λ sufficiently negative, such a problem admits $2^n - 1$ positive solutions. This kind of multiplicity was then proved when $\lambda = 0$ and the negative part of the weight is sufficiently large in [12], by using a shooting technique (we also mention that the same results have been obtained for large solutions in [8]), and, later, in [5], in the PDE case by means of variational methods. Recently, in [10], the use of topological degree has allowed the authors to obtain the same kind of results for more general nonlinearities and $\lambda \sim 0$.

In the case of Neumann boundary conditions, analogous results have been obtained in [7] with shooting techniques and in [11] with the coincidence degree. This similarity in the behavior of superlinear indefinite problem with Dirichlet and Neumann boundary conditions, arises the natural question of whether high multiplicity results in the spirit of [19] can be obtained with the simple weight function of (1.2), which has a unique positive component (observe that in such a case, both for Dirichlet and Neumann boundary conditions the results of [12,7,10,11] guarantee the existence of just $2^1 - 1 = 1$ positive solution).

In this work we will positively answer this question by using the same topological shooting technique of [19], which firstly consists in studying separately the sets of points reached in the phase plane by all the solutions of the problem in $(0, \alpha)$ and $(1-\alpha, 1)$, i.e. where the weight is negative, and then in connecting such sets through the flow in $(\alpha, 1-\alpha)$, where the weight is positive. For this last point, we perform a careful analysis of the time maps that allow us to establish all the types of connections.

Here, however, contrarily to [19], we consider homogeneous boundary conditions, which requires a sharper analysis of the solutions of the sublinear parts near $u = 0$ (see Theorem 2.2(iv)). In addition, this makes not clear how to find particular values of b to get high multiplicity, as it was the case in [19], and then let b vary to obtain the structure of the bifurcation diagrams.

To overcome this problem, we use a new approach that consists in using α as the main bifurcation parameter, regulating in this way the amplitude of the region in which the weight is positive. Firstly, we obtain arbitrarily high multiplicity, when λ is sufficiently negative, for the purely superlinear problem corresponding to $\alpha = 0$ (we point out that similar results have been recently obtained with different