J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Maximum principle in some unbounded domains for an elliptic operator with unbounded drift ☆

Sungwon Cho

Department of Mathematics Education, Gwangju National University of Education, 55 Pilmundaero Buk-gu, Gwangju 61204, Republic of Korea

A R T I C L E I N F O

Article history:
Received 21 March 2018
Available online xxxx
Submitted by M. Musso

Keywords:
Maximum principle
Second order elliptic equation
Measurable coefficients
Unbounded domain

A B S T R A C T

We study the maximum principle for (wG)-domain, which can be unbounded. The (wG)-condition was introduced by A. Vitolo and the maximum principle was proved for the uniformly elliptic operator with bounded coefficients. Using Safonov's growth lemma, we treat the operator when the first order coefficients belong to n-integrable Lebesgue space. Some examples including infinite cones, various unbounded coefficients, are presented.

1. Introduction

We study the following (weak) maximum principle:

Definition 1.1 *(See [3, Definition on page 47]).* We say that the (weak) maximum principle does hold for the operator L in Ω if

$$Lu \geq 0 \quad \text{in } \Omega,$$

and

$$\limsup_{x \to \partial\Omega} u(x) \leq 0$$

imply $u \leq 0$ in Ω.

☆ Funding: This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2017R1D1A1B03028258).

E-mail address: scho@gnue.ac.kr.

https://doi.org/10.1016/j.jmaa.2018.07.056

Especially, we are interested in unbounded domains. In fact, many researchers contributed to the subject in various settings. See for instance, A. Bonfiglioli and E. Lanconelli [4] for a sub-Laplacian operator, S. Pigola, M. Rigoli and A. Setti [17,15,16] for manifolds, J. Busca [5], and I. Capuzzo Dolcetta, F. Leoni and A. Vitolo [8] for fully nonlinear equations and viscosity solutions, I. Birindelli, I. Capuzzo Dolcetta and A. Vitolo [1], and I. Capuzzo Dolcetta and A. Vitolo [10] for singular or degenerate elliptic operators.

In this paper, we consider the following form of the operator L:

$$L = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} + \sum_{i=1}^{n} b_i(x) D_i + c(x), \tag{L}$$

where D_i represents a partial derivative with x_i-direction, namely

$$D_i = \frac{\partial}{\partial x_i}, \quad \text{and } D_{ij} = D_i D_j = \frac{\partial^2}{\partial x_j \partial x_i}.$$

Here, we assume that the coefficients a_{ij}, b_i, c are measurable functions, not necessarily continuous, and L is a uniformly elliptic operator. A uniform ellipticity implies that by definition, a_{ij} satisfy, for some strictly positive constants λ, Λ,

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n) \tag{UE}$$

for any $x \in \Omega(\subset \mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$, for a given domain, open and connected set Ω in $\mathbb{R}^n$, $n \ge 2$. Thus, the operator L is a second order uniformly elliptic linear differential operator of nondivergence form.

Throughout the paper, we assume that

$$a_{ij}(x) = a_{ji}(x) \quad \text{for each } i, j = 1, 2, ..., n, \quad c \le 0. \tag{1}$$

Unless stated otherwise, we also assume that the operator L acts on bounded above functions in the Sobolev function space $W^{2,n}_{loc}(\Omega)$, which means that $u_+ \in L^\infty(\Omega), u, D_i u, D_{ij}u \in L^n(\Omega')$, the n-integrable Lebesgue space for any bounded open set $\Omega' \subset \overline{\Omega'} \subset \Omega$.

One can easily show the maximum principle in a bounded domain via the classical Aleksandrov–Bakel'man–Pucci estimate (ABP estimate, in short):

$$\sup_\Omega u \le \limsup_{x\to\partial\Omega} u_+ + C \cdot diam(\Omega) \cdot \|f_-\|_{L^n(\Omega)},$$

whenever $Lu \ge f$. Here, C is a constant depending only on λ, Λ, n, and $u_+ := max\{u, 0\}$, $f_- := max\{-f, 0\}$, $diam(\Omega)$ is a diameter of the domain Ω. For the proof and more details, one may refer to [2,11]. Using ABP estimate, we have

$$\sup_\Omega u \le \limsup_{x\to\partial\Omega} u_+ \le 0$$

provided that $Lu \ge 0$ and $\limsup_{x\to\partial\Omega} u(x) \le 0$.

But, in general the maximum principle does not hold in unbounded domains as illustrated in the following examples:

Half space $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n | x_n > 0, x = (x_1, ..., x_n)\}$.

Consider L is the Laplace operator of $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. Then the linear function $u(x) = x_n$ is harmonic in $\mathbb{R}^n_+$ and $u = 0$ on $\partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_n = 0, x = (x_1, ..., x_n)\}$, but is strictly positive on its interior. In this case, the function u is not convergent nor bounded for large $|x|$.

Please cite this article in press as: S. Cho, Maximum principle in some unbounded domains for an elliptic operator with unbounded drift, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.07.056