J. Math. Anal. Appl. 467 (2018) 734–750

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Stability of the overdamped Langevin equation in double-well potential

Nikola Sandrić

Department of Mathematics, University of Zagreb, 10000 Zagreb, Croatia

ARTICLE INFO

Article history:
Received 9 January 2018
Available online 26 July 2018
Submitted by U. Stadtmueller

Keywords:
Diffusion process
Langevin equation
Double-well potential
Stability

ABSTRACT

In this article, we discuss stability of the one-dimensional overdamped Langevin equation in double-well potential. We determine unstable and stable equilibria, and discuss the rate of convergence to stable ones. Also, we derive conditions for stability of general diffusion processes which generalize the classical and well-known results of Khasminskii ([8]).

1. Introduction

The *Langevin equation* is a stochastic differential equation describing the dynamics of a particle immersed in a fluid, subjected to an external potential force field and collisions with the molecules of the fluid:

$$\begin{aligned} m\,\mathrm{d}X_t &= P_t\mathrm{d}t \\ \mathrm{d}P_t &= -(\lambda/m)P_t\mathrm{d}t - \nabla V(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad (X_0,P_0)\in\mathbb{R}^d\times\mathbb{R}^d. \end{aligned} \tag{1.1}$$

Here, $\{X_t\}_{t\ge0}$ and $\{P_t\}_{t\ge0}$ denote, respectively, the position and momentum of the particle, m is particle's mass, $-(\lambda/m)P_t\mathrm{d}t$, $\lambda>0$, is the velocity-proportional damping (friction) force, V is particle's potential and $\sigma(X_t)\mathrm{d}B_t$ is the noise term representing the effect of the collisions with the molecules of the fluid, where $\{B_t\}_{t\ge0}$ denotes a standard d-dimensional Brownian motion. Observe that here we assume the measure of the noise strength σ is non-constant, meaning that the effect of collisions depends on the position of the particle (e.g. due to heterogeneity of the fluid). In this case, the function σ models the nature of the position-dependence.

E-mail address: nsandric@math.hr.

https://doi.org/10.1016/j.jmaa.2018.07.043

In the case when the inertia of the particle is negligible in comparison with the damping force (due to friction), the trajectory of the particle is described by the so-called *overdamped Langevin equation*:

$$\lambda \,\mathrm{d}X_t = -\nabla V(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad X_0 \in \mathbb{R}^d. \tag{1.2}$$

Namely, in [16, Chapter 10] it has been shown that (under certain assumptions on the potential V and diffusion coefficient σ) the solution to (1.1) converges a.s. to the solution to (1.2), as $m \searrow 0$.

The main purpose of this article is to discuss stability of the solution to the one-dimensional overdamped Langevin equation in *double-well* or *Landau potential* $V(x) = -ax^2/2 + bx^4/4$, $a, b > 0$:

$$\lambda \,\mathrm{d}X_t = (-bX_t^3 + aX_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \qquad X_0 \in \mathbb{R}. \tag{1.3}$$

This potential is of considerable interest in quantum mechanics and quantum field theory for the exploration of various physical phenomena or mathematical properties since it permits in many cases explicit calculation without over-simplification (see e.g. [3] and [11]). Typical example where it occurs is in the so-called *ammonia inversion phenomenon*. This is a switching of the nitrogen atom from above to below the hydrogen plane. More precisely, the ammonia molecule is pyramidal shaped with the three hydrogen atoms forming the base and the nitrogen atom at the top. The nitrogen atom sees a double-well potential with one well on either side of the hydrogen plane. Because the potential barrier is finite, it is possible for the nitrogen atom to tunnel through the plane of the hydrogen atoms, thus "inverting" the molecule (see [10] for more details).

For the sake of simplicity, but without loss of generality, in the sequel we assume $a = b = \lambda = 1$. Also, we impose the following assumptions on the diffusion coefficient σ:

A1 σ is locally Lipschitz continuous;

A2 $\limsup_{|x|\nearrow\infty} |\sigma(x)|/|x|^2 < \sqrt{2}$.

Under (**A1**) and (**A2**), in [2, Theorem 3.1 and Proposition 4.2] and [17, Theorem 3.1.1] it has been shown that the equation in (1.3) admits a unique non-explosive strong solution $\{X_t\}_{t\geq 0}$ which, in addition, is a temporally homogeneous strong Markov process with continuous sample paths. Furthermore, in [2, Remark 2.2 and Proposition 4.3] it has been also shown that $\{X_t\}_{t\geq 0}$ is a $\mathcal{C}_b$-Feller process and that for any $f \in \mathcal{C}^2(\mathbb{R})$ the process

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\mathrm{d}s, \qquad t \geq 0, \tag{1.4}$$

is a local martingale, where

$$\mathcal{L}f(x) = (-x^3 + x)f'(x) + \frac{\sigma^2(x)}{2}f''(x), \qquad f \in \mathcal{C}^2(\mathbb{R}). \tag{1.5}$$

Recall, $\mathcal{C}_b$-*Feller property* means that the semigroup $\{P_t\}_{t\geq 0}$ of $\{X_t\}_{t\geq 0}$, defined as

$$P_t f(x) := \int_{\mathbb{R}} f(y)p^t(x, \mathrm{d}y), \qquad t \geq 0,\ x \in \mathbb{R},\ f \in \mathcal{B}_b(\mathbb{R}),$$

maps $\mathcal{C}_b(\mathbb{R}) := \mathcal{C}(\mathbb{R}) \cap \mathcal{B}_b(\mathbb{R})$ to $\mathcal{C}_b(\mathbb{R})$. Here, $p^t(x, \mathrm{d}y)$ and $\mathcal{B}_b(\mathbb{R})$ denote, respectively, the transition kernel of $\{X_t\}_{t\geq 0}$ and the space of bounded Borel measurable functions.