Doctopic: Miscellaneous ARTICLE IN PRESS YJMAA:22312

J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On abstract differential equations with state dependent non-local conditions

Eduardo Hernández [1]

Departamento de Computação e Matemática, Faculdade de Filosofia Ciências e Letras de Ribeirão Preto Universidade de São Paulo, CEP 14040-901 Ribeirão Preto, SP, Brazil

A R T I C L E I N F O

Article history:
Received 2 February 2018
Available online xxxx
Submitted by J. Shi

Keywords:
State dependent delay equations
Strict solutions
Analytic semigroup
Partial differential equations

A B S T R A C T

In this paper we introduce the concept of "state dependent nonlocal condition" and study the existence and uniqueness of solution for a general class of abstract differential equations with state dependent delay subject to this class of conditions. In this type of problem the nonlocal condition can be dependent on times or time intervals determined by the (unknown) solution. Some examples on partial differential equations with state dependent delay arising in population dynamics are presented.

1. Introduction

In this work we introduce the concept of "state dependent nonlocal condition" and study the existence and uniqueness of solution for a general class of abstract delayed differential equation subject to this class of conditions. Specifically, we study the existence and uniqueness of mild and strict solutions for a class of differential equations of the form

$$u'(t) = Au(t) + F(t, u_{\eta(t,u_t)}), \quad t \in [0, a], \tag{1.1}$$

$$u_0 = G(\sigma(u), u) \in \mathcal{B}_X = C([-p, 0]; X), \tag{1.2}$$

where $A : D(A) \subset X \mapsto X$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|)$, $F \in C([0, a] \times \mathcal{B}_Z; W)$, $\eta \in C([0, a] \times \mathcal{B}_Z; [0, a])$, $\sigma \in C(C([-p, a]; Z); [0, a])$, $G \in C([0, a] \times C([-p, a]; Z); \mathcal{B}_W)$ are function to be specified later, $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ are Banach spaces and $\mathcal{B}_S = C([-p, 0]; S)$ for $S = W, Z$.

The nonlocal condition (1.2), called "state dependent nonlocal condition", is motivated by theory and applications, and generalize several types of nonlocal conditions studied in the literature. This condition

E-mail address: lalohm@ffclrp.usp.br.

[1] The work of this author is supported by Fapesp 2017/13145-8.

https://doi.org/10.1016/j.jmaa.2018.05.080

can be seen as a feedback control by which an qualitative property or magnitude of the unknown solution is used as initial state. The condition $u_0 = u_{\sigma(u)}$, where $\sigma \in C(C([-p,a];X);[0,a])$ is a function defined by an equation of the form $\int_{-p}^{\sigma(u)} K(s,u(s,\cdot))ds = Q$ (a threshold condition, see [22,29,30]), is an example of the type of nonlocal conditions considered in this work.

The problem of existence and uniqueness of solutions of (1.1)–(1.2) is non-trivial since functions of the form $u \mapsto u_{\sigma(u)}$ are (in general) nonlinear and non-Lipschitz in spaces of continuous functions. Noting that

$$\|u_{\sigma(u)} - v_{\sigma(v)}\|_{\mathcal{B}_X} \leq (1 + [v]_{C_{Lip}([-p,a],X)}[\sigma]_{C_{Lip}(C([-p,a];X);[0,a])}\|u - v\|_{C([-p,a];X)},$$

when the involved functions are Lipschitz, we can study this problem on spaces of Lipschitz functions, a hard approach in the semigroup framework since require estimates for $[u_{\sigma(u)} - v_{\sigma(v)}]_{C_{Lip}([-p,a],X)}$ and $[F(\cdot, u_{\eta(\cdot,u_{(\cdot)})}) - F(\cdot, v_{\eta(\cdot,v_{(\cdot)})})]_{C_{Lip}([0,a],X)}$, which usually depends on $u'(\cdot)$, see [25] for additional details. To overcome this difficulty, we use the fact that sets of the form $\{u \in C([-p,a];X) : [u]_{C_{Lip}([-p,a];X)} \leq R\}$ (with $R > 0$) are closed in $C([-p,a];X)$.

The study of differential equations with nonlocal conditions has been stimulated by theory and applications. Concerning the literature, we cite the pioneers works by Byszewski et al. [13,12] for ODEs, the monographs by Burlica et al. [11] and the articles [2,5,4,3,14,44,26,34,35] for partial and abstract differential equations without delay. For the case of abstract problems with delay, we cite the recent interesting papers by Burlica et al. [11,10,9] and Vrabie et al. [6,41,40].

The theory of state-dependent delay differential equations is a field of intense research because applications and the fact that the qualitative theory is different from the theories of differential equations with discrete and time dependent delay. Concerning ODEs on finite dimensional spaces we cite the early articles by Driver [15,16], Mackey & Glass [33] and Aiello, Freedman & Wu [1], the survey by Hartung, Krisztin, Walther & Wu [22], the papers [20,21,42,43] and the references therein. For first order abstract differential equations with applications to partial differential equations, we refer the reader to [24,27,38,39]. We also cite the recent and interesting articles by Krisztin & Rezounenko [29], Yunfei, Yuan & Pei [32], Kosovalic, Chen & Wu [28] and Hernandez, Pierri & Wu [25].

For problems with nonlocal conditions similar to (1.2), we just cite [23]. In comparison to [23], we generalize the theory for abstract differential equations with state-dependent delay and present several new results on the existence of mild and strict solution.

Next, we note some of the results in paper. The Theorem 2.1, Theorem 2.2 and Proposition 2.1 prove the existence and uniqueness of mild and strict solutions supposing that $G(\cdot)$, $F(\cdot)$, $\sigma(\cdot)$ and $\eta(\cdot)$ are Lipschitz, $G(\cdot)$ and $F(\cdot)$ are bounded and some conditions guaranteeing that $T(\cdot)G(\sigma(u),u)(0)$ is Lipschitz. The statement of Proposition 2.1 is very general permitting to consider different situations and applications (see, for instance, Corollary 2.3 and Example (3.3)–(3.4)). The case where $G(\cdot)$ and $F(\cdot)$ are unbounded and (or) locally Lipschitz is treated in Proposition 2.2, Proposition 2.3 and Proposition 2.4. The Theorem 2.3 and Proposition 2.5 provide the existence of a solution using the Schauder's fixed point theorem. The existence of solutions for problems defined on $[0,\infty)$ is studied in Section 2.1. In the last section, we use our results to study some examples considered in studies on state dependent delay equations.

We note that our results on the existence and uniqueness of mild solutions can be proved without assume that the semigroup is analytic. The choice to use analytic semigroups is because of applications to nonlinear problems (see Corollary 2.3 and Example (3.3)–(3.4)) and our interest in strict solutions.

Next, we include some notations and results. Let $(Z, \|\cdot\|_Z)$, $(W, \|\cdot\|_W)$ be Banach spaces, $I \subset \mathbb{R}$ be an interval and $l > 0$. Next, $\mathcal{B}_Z$ is the space $C([-p,0];Z)$ endowed with the uniform norm denoted by $\|\cdot\|_{\mathcal{B}_Z}$, $B_l(z,Z) = \{x \in Z; \|x - z\|_Z \leq r\}$ and $\mathcal{L}(Z,W)$ is the space of bounded linear operators from Z into W endowed with the operator norm denoted by $\|\cdot\|_{\mathcal{L}(Z,W)}$. We write $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ if $Z = W$ and simply $\|\cdot\|$ when $W = X$. The spaces $C(I;Z)$ and $C_{Lip}(I;Z)$ are the usual and their norms are denoted by $\|\cdot\|_{C(I;Z)}$ and $\|\cdot\|_{C_{Lip}(I;Z)}$, respectively. We remark that $\|\cdot\|_{C_{Lip}(I;Z)} = \|\cdot\|_{C(I;Z)} + [\cdot]_{C_{Lip}(I;Z)}$

Please cite this article in press as: E. Hernández, On abstract differential equations with state dependent non-local conditions, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.05.080