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Algebraic traveling waves for the generalized viscous Burgers equation

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ABSTRACT

In this paper we apply the general results in [10] on algebraic traveling wave solutions for n-th order nonlinear evolution equations in one space dimension to prove a full classification of the algebraic traveling wave solutions of a second order generalized viscous Burgers' equation.

1. Introduction and statement of the main results

Nonlinear partial differential equations have been the subject of study in various branches of mathematical–physical sciences such as physics, biology, chemistry, etc. The analytical solutions of such equations are of fundamental importance because mostly of the natural phenomena are modelized mathematically by non-linear partial differential equations. Among the possible solutions, there are ones of a certain special form which have been widely studied. These are the so-called traveling wave solutions that are solutions which do not change their shape, and that propagate at constant speed. These waves appear in fluid dynamics [18,20], chemical kinetics involving reactions [11,21], mathematical biology [12,24], lattice vibrations and solid state physics [23], plasma physics and laser theory [16], optical fibers [12],...

There are various approaches for constructing traveling wave solutions. Some of these approaches are the Jacobi elliptic function method [22], inverse scattering method [1], Hirotas bilinear method [15], homogeneous balance method [27], homotopy perturbation method [14], Weierstrass function method [28], symmetry method [5], Adomian decomposition method [3], sine/cosine method [6], tan/coth method [17], the Exp-function method [7], etc. But most of the methods may sometimes fail or can only lead to a kind of special solution and the solution procedures become very complex as the degree of the nonlinearity increases. In [10] the authors gave a necessary and sufficient condition for a partial differential equation to have

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explicit algebraic traveling waves. More precisely, they gave a technique to prove the existence of traveling wave solutions for general n-th order partial differential equations by showing that traveling wave solutions exist if and only if the associated n-dimensional first order ordinary differential equation has some invariant algebraic curve. In this paper we will consider only the case $n = 2$.

More precisely, consider the second order partial differential equations of the form

$$\frac{\partial^2 u}{\partial x^2} = F\Big(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\Big), \tag{1}$$

where x and t are real variables and F is a smooth map. The traveling wave solutions of system (1) are particular solutions of the form $u = u(x,t) = U(x - ct)$ where $U(s)$ satisfies the boundary conditions

$$\lim_{s\to-\infty} U(s) = a \quad \text{and} \quad \lim_{s\to\infty} U(s) = b, \tag{2}$$

where a and b are solutions, not necessarily different, of $F(u, 0, \dots, 0) = 0$. Plugging $u(x,t) = U(x-ct)$ into (1) we get that $U(s)$ has to be a solution, defined for all $s \in \mathbb{R}$, of the second order ordinary differential equation

$$U^{(2)} = F(U, U', -cU), \tag{3}$$

where $U(s)$ and the derivatives are taken with respect to s. The parameter c is called the *wave speed* of the traveling wave solution.

We say that $u(x,t) = U(x - ct)$ is an algebraic traveling wave solution if $U(s)$ is a nonconstant function that satisfies (2) and (3) and there exists a polynomial $p \in \mathbb{R}[z,w]$ such that $p(U(s), U'(s)) = 0$. We note that in the definition of traveling wave solutions we exclude periodic traveling wave solutions.

The main result that we will use is the following theorem, see [10] for its proof.

Theorem 1. *The partial differential equation (1) has an algebraic traveling wave solution with respect to c if and only if the first order differential system (4)*

$$\begin{cases} y_1' &= y_2, \\ y_2' &= G_c(y_1,y_2), \quad G_c(y_1,y_2) = F(y_1,y_2) \end{cases} \tag{4}$$

has an invariant algebraic curve containing the critical points $(a, 0)$ and $(b, 0)$ and no other critical points between them.

The main result of this paper is, with the techniques in [10], find all explicit algebraic traveling wave solutions of the generalized viscous Burgers equation

$$\frac{\partial u}{\partial t} + bu\frac{\partial u}{\partial x} - a\frac{\partial^2 u}{\partial x^2} = du(1-u) \tag{5}$$

where $a, b, d \in \mathbb{R}$ with $a \neq 0$. Note that system (5) contains both the viscous Burgers's equation (when $b = 1$, $d = 0$), and the Fisher–Kolmogorov equation (when $b = 0$, $a = d = 1$). The term $a\partial^2 u/\partial x^2$ is a diffusion term. The term $bu\partial u/\partial x$ has a shocking up effect that causes waves to break (when $d = 0$) and the term $du(1-u)$ is the nonlinear term that takes into account the reaction of a process (that is, it is a growth term) and it will affect the velocity speed of the wave.

In the following theorem we provide a full characterization of all algebraic traveling wave solutions of the generalized viscous Burgers equation (5).