J. Math. Anal. Appl. 466 (2018) 523–566

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Limit theorems and the support of SDES with oblique reflections on nonsmooth domains ☆

Jing Wu, Mingbo Zhang *

School of Mathematics, Sun Yat-sen University Guangzhou, Guangdong 510275, PR China

ARTICLE INFO

Article history:
Received 12 November 2017
Available online 5 June 2018
Submitted by U. Stadtmueller

Keywords:
Wong–Zakai approximation
Stochastic differential equations
Oblique reflection
Approximate continuity
Support

ABSTRACT

In this paper we consider the Wong–Zakai type approximation and the support problem of stochastic differential equations with oblique reflections in nonsmooth domains satisfying conditions introduced by Dupuis and Ishii [4]. We first establish the limit theorem for reflected diffusions by proving that the sequence of solutions of approximating equations converge uniformly in L^p to the unique solution of the reflected stochastic differential equation. We also prove the approximate continuity and thus characterize the support of diffusions described by reflected SDEs.

1. Introduction

The classical relation between stochastic differential equations (SDEs in short) and deterministic ordinary differential equations was first found by Wong and Zakai [11], and got extensively studied by many following researchers. For SDEs with normal reflection in smooth domains, Doss and Priouret [2] have established the convergence in probability of the Wong–Zakai approximation. The case of normal reflections in convex domains has been studied in [8], where the diffusion coefficients are constants. For the more general case with nonconstant diffusions, the limit theorem has been established in [10], where Ren and Xu applied the convergence result to describe the support of solutions of stochastic variational inequalities. For the case reflected in domains neither smooth nor convex, Aida–Sasaki [1] considered the Wong–Zakai approximation and obtained the limit theorem.

In the present paper we are concerned with the Wong–Zakai approximation of SDEs with oblique reflections in a nonsmooth and non-convex domain $D \subset \mathbb{R}^d$:

☆ Work supported by NSFC (No. 11671408, 11471340) and Pearl River Nova Program of Guangzhou (201710010045).

* Corresponding author.
E-mail addresses: wujing38@mail.sysu.edu.cn (J. Wu), zhangmb@mail2.sysu.edu.cn (M. Zhang).

https://doi.org/10.1016/j.jmaa.2018.06.004
0022-247X/

$$X(t) = x + \int_0^t \sigma(X(s))dB(s) + \int_0^t b(X(s))ds + \Phi(t), \qquad x \in \bar{D}. \tag{1.1}$$

Existence and uniqueness results of solutions are proved in [4] when D satisfies the following conditions in **Case 1** or **Case 2**:

Case 1. The set of directions of reflection is actually a singleton: $R(x) = \{\gamma(x)\}$, and $\gamma \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d)$. There exists a $b \in (0,1)$ such that

$$\cup_{0 \leqslant t \leqslant b} B(x - t\gamma(x), tb) \subset D^c, \tag{1.2}$$

for $x \in \partial D$.

Case 2. $D = \cap_{i=1}^m D_i$ for some $m \geqslant 1$ and each D_i is an open bounded set. Let $I(x) = \{i : x \notin D_i\}$. For every $x \in \partial D$, there is an open neighborhood V of x such that for $y \in V$, $I(y) \subset I(x)$.

- For each $i = 1, \cdots, m$, ∂D_i is of class $\mathcal{C}^1$.
- Assume for each i, there exists $\gamma_i \in \mathcal{C}^{0,1}(\mathbb{R}^d, \mathbb{R}^d)$ such that if $n_i(x)$ denotes the unit inward normal vector at $x \in \partial D_i$, $\langle n_i(x), \gamma_i(x) \rangle > 0$. Assume also for every $x \in \partial D$ the convex hull of $\{\gamma_i(x) : i \in I(x)\}$ does not contain the origin. Define the directions of reflection as

$$R(x) = \left\{ \sum_{i \in I(x)} c_i \gamma_i(x) : c_i \geqslant 0, \left| \sum_{i \in I(x)} c_i \gamma_i(x) \right| = 1 \right\}. \tag{1.3}$$

  Then the set $\{\alpha\gamma : \alpha \geqslant 0, \gamma \in R(x)\}$ is a closed convex cone and there is a hyperplane through the origin that intersects this cone only at the origin.
- For each $x \in \partial D$, $\exists \alpha_i \geqslant 0,\ i \in I(x)$ such that

$$\langle \sum_{i \in I(x)} \alpha_i \gamma_i(x), n_j(x) \rangle > 0 \quad for \quad j \in I(x). \tag{1.4}$$

- For every $x \in \partial D$ there is an open neighborhood Q of x and a family $\{D(y) : y \in Q\}$ of compact convex subsets of $\mathbb{R}^d$ with $0 \in D(y)$ for all $y \in Q$ such that the function $(y, x) \to d(x, D(y))^2$ is $\mathcal{C}^{2,+}(Q \times \mathbb{R}^d)$ and that for all $y \in Q \cap \partial D$, $p \in \partial D(y)$, for any inward normal vector n at p,

$$\langle \gamma_i(y), n \rangle \begin{cases} \geqslant 0, & \text{if } \langle p, n_i(y) \rangle \geqslant -1 \\ \leqslant 0, & \text{if} \langle p, n_i(y) \rangle \leqslant 1. \end{cases} \tag{1.5}$$

Here $B(a, r) = \{x \in \mathbb{R}^d : |x - a| \leqslant r\}$, $S(a, r) = \{x \in \mathbb{R}^d : |x - a| = r\}$, $d(x, O) = \inf\{|x - y| : y \in O\}$, and $\mathcal{C}^{2,+}(O)$ is the set of real, locally semiconcave functions on O.

For $0 < \theta \leqslant 1$, set

$$\|\omega\|_{\mathcal{H},[s,t],\theta} := \sup_{s \leqslant u \leqslant v \leqslant t} \frac{|\omega(u) - \omega(v)|}{|u - v|^\theta},$$
$$\|\omega\|_{\infty,[s,t]} = \max_{s \leqslant u \leqslant v \leqslant t} |\omega(u) - \omega(v)|,$$
$$\|\omega\|_{[s,t]} = \sup_\Delta \sum_{k=1}^N |\omega(t_k) - \omega(t_{k-1})|,$$

where $\Delta = s = t_0 < \ldots < t_N = t$ is a partition of the interval $[0, t]$. Denote by $D_x f(x, y)$ and $D_y f(x, y)$ respectively the gradients of $f(x, y)$ with respect to x and y. Let $\mathbb{S}^d$ denote the set of all symmetric $d \times d$ matrices. For a function f defined on $U \subset \mathbb{R}^d$, define