Doctopic: Functional Analysis ARTICLE IN PRESS YJMAA:22318

J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Multinormed semifinite von Neumann algebras, unbounded operators and conditional expectations

Anar Dosi

Middle East Technical University Northern Cyprus Campus, Guzelyurt, KKTC, Mersin 10, Turkey

ARTICLE INFO

Article history:
Received 29 January 2018
Available online xxxx
Submitted by M. Mathieu

Keywords:
Multinormed W^*-algebra
Unbounded operators
Central topology
ℓ_1-normed space

ABSTRACT

The present paper is devoted to classification of multinormed W^*-algebras in terms of their bounded parts. We obtain a precise description of the bornological predual of a multinormed W^*-algebra, which is reduced to a multinormed noncommutative L^1-space in the semifinite case. A multinormed noncommutative L^2-space is obtained as the union space of a commutative domain in a semifinite von Neumann algebra. Multinormed W^*algebras of type I are described as locally bounded decomposable (unbounded) operators associated with a measurable covering.

1. Introduction

The projective limits of von Neumann algebras with their normal connecting homomorphisms (called normal projective limits) play a key role in many problems of noncommutative probability theory, quantum groups and noncommutative geometry. Noncommutative Sobolev spaces are defined in [20] by M. Mizuo as the inverse limits of von Neumann algebras generated by a semicircular system based on the viewpoints of noncommutative topology and noncommutative probability theory. A possible candidate for the algebra $L^\infty(\mathbb{S}_\infty)$ over the quantum semigroup $\mathbb{S}_\infty$ of all quantum permutations of a countable set was suggested in [14] by D. Goswami and A. Skalski to be a projective limit of the enveloping von Neumann algebras of the C^*-algebras $C(\mathbb{S}_n)$ of quantum permutations of n elements. The algebra (actually module) of noncommutative martingales turns out to be a projective limit of an increasing family of von Neumann algebras whose connecting maps are conditional expectations [27]. In the same framework the space of martingales is defined as inverse limit of the Gelfand–Tsetlin algebras GZ_n which are invariant under the given conditional expectations [27]. The normal projective limits of von Neumann algebras named as multinormed W^*-algebras in [10] were investigated first in [12] by M. Fragoulopoulou (see also [19]). The representation theorem from [12] (see also [10]) asserts that each multinormed W^*-algebra is topologically $*$-isomorphic to

E-mail address: dosiev@yahoo.com.
URL: http://webwork.ncc.metu.edu.tr/members/dosi.php.

https://doi.org/10.1016/j.jmaa.2018.06.002
0022-247X/© 2018 Elsevier Inc. All rights reserved.

a strongly closed $*$-subalgebra of the O*-algebra $C^*_{\mathcal{E}}(\mathcal{D})$ of all noncommutative continuous functions on a domain $\mathcal{E}$ with its union space $\mathcal{D}$.

Recall that an O*-algebra (see [23]) is treated as a unital $*$-algebra A of linear transformations on a common dense subspace $\mathcal{D}$ of a Hilbert space H which leaves invariant $\mathcal{D}$. Each $a \in A$ as an unbounded operator on H admits unbounded dual $a^{\bigstar}$ which leaves invariant $\mathcal{D}$, therefore $a^* = a^{\bigstar}|\mathcal{D} \in A$ and the correspondence $a \mapsto a^*$ turns out to be a natural involution on A. A domain $\mathcal{E}$ in H with its union space $\mathcal{D}$ is defined as an upward filtered family of commuting projections $\mathcal{E}$ in $\mathcal{B}(H)$ such that $\vee\mathcal{E} = 1$ and $\mathcal{D} = \cup\mathcal{E}(H)$, where $\vee\mathcal{E}$ is the least upper bound in $\mathcal{B}(H)$ of the projection net. The multinormed W^*-algebra $C^*_{\mathcal{E}}(\mathcal{D})$ of all noncommutative continuous functions on a domain $\mathcal{E}$ is defined as the O*-algebra of those linear transformations x on $\mathcal{D}$ such that $ex \subseteq xe$ and $xe \in \mathcal{B}(H)$ for all $e \in \mathcal{E}$ (see [5], [6], [8] and [1]). The family of C^*-seminorms $\|x\|_e = \|xe\|$, $x \in C^*_{\mathcal{E}}(\mathcal{D})$, $e \in \mathcal{E}$ defines the original polynormed (or locally convex) topology of $C^*_{\mathcal{E}}(\mathcal{D})$, whereas the seminorms $p_{\zeta,\eta}(x) = |(x\zeta,\eta)|$, $\zeta,\eta \in \mathcal{D}$, and $p_\zeta(x) = \|x\zeta\|$, $\zeta \in \mathcal{D}$, define the weak (WOT) and strong (SOT) operator topologies in $C^*_{\mathcal{E}}(\mathcal{D})$ (see [23, 3.3], [18, Ch. I]), respectively. One of the central results of [10] asserts that every multinormed W^*-algebra turns out to be a central completion of a von Neumann algebra, which in turn is identified with SOT-closed $*$-subalgebra of $C^*_{\mathcal{E}}(\mathcal{D})$ up to a $*$-isometry of the multinormed algebras. Namely, let $\mathcal{M}$ be a von Neumann algebra with a domain $\mathcal{E}$ in $\mathcal{M}$, that is, $\mathcal{E}$ is a family of central projections in $\mathcal{M}$ such that $\vee\mathcal{E} = 1$. The multinormed W^*-algebra $\mathcal{M}_{\mathcal{E}}$ is defined as the polynormed completion of $\mathcal{M}$ (called its bounded part) equipped with the polynormed topology defined by the family of C^*-seminorms $\|x\|_e = \|xe\|$, $x \in \mathcal{M}$ called *a central topology of* $\mathcal{M}$. Moreover, $\mathcal{M}_{\mathcal{E}}$ admits a unique (up the isometry) predual X (in the bornological sense), which is an ℓ_1-normed space (see below Definition 3.1), and $\mathcal{M}_{\mathcal{E}} \subseteq C^*_{\mathcal{E}}(\mathcal{D})$ up to a $*$-isometry of the multinormed algebras. Thus $C^*_{\mathcal{E}}(\mathcal{D})$ plays the role of locally convex $\mathcal{B}(H)$.

The present paper is devoted to the characterizations of multinormed W^*-algebras in terms of their bounded parts. We say that $\mathcal{M}_{\mathcal{E}}$ is *a multinormed semifinite* (resp., *finite*) *W^*-algebra* if it a normal inverse limit of semifinite (resp., finite) von Neumann algebras. Similarly, we say that $\mathcal{M}_{\mathcal{E}}$ is of certain type if it a normal inverse limit of von Neumann algebras of the same type. It is not hard to prove that a multinormed W^*-algebra $\mathcal{M}_{\mathcal{E}}$ is of certain type iff its bounded part $\mathcal{M}$ has the same type. In particular, $\mathcal{M}_{\mathcal{E}}$ is a multinormed semifinite (resp., finite) W^*-algebra iff $\mathcal{M}$ is a semifinite (resp., finite) von Neumann algebra (see below Lemma 5.1). We classify multinormed semifinite W^*-algebras using continuous extensions of faithful semifinite normal traces τ from their bounded parts. The framework suggested allows to construct the multinormed noncommutative L^2-space $L^2(\mathcal{M},\tau)_{\mathcal{E}}$ to be the union space of the isomorphic copy of the domain $\mathcal{E}$. The latter justifies our choice of the union space $\mathcal{D}$ of the domain $\mathcal{E}$ in the theory of multinormed W^*-algebras. Similarly, we construct the multinormed noncommutative L^1-space $L^1(\mathcal{M},\tau)_{\mathcal{E}}$ to be ℓ^1-normed predual of $\mathcal{M}_{\mathcal{E}}$. The conditional expectations occur as contractive (in the locally convex sense) projections that explains injectivity property of quantum spaces investigated in [11]. In section 3 we propose an explicit expression for the (bornological) predual of the multinormed W^*-algebra $\mathcal{M}_{\mathcal{E}}$. If $\mathcal{M}_*$ is the predual of the von Neumann algebra $\mathcal{M}$ then $\mathcal{M}_*$ possesses a natural $\mathcal{M}$-bimodule structure, and we put $\mathcal{M}_{*\mathcal{E}} = \sum_{e\in\mathcal{E}} \mathcal{M}_* e$, which is a dense subspace of $\mathcal{M}_*$. The normed space $\mathcal{M}_{*\mathcal{E}}$ equipped with the bornology $\{\text{ball}\,\mathcal{M}_* e\}$ is an ℓ^1-normed space. The $\mathcal{M}$-bimodule structure on $\mathcal{M}$ defines the linear mapping $T : \mathcal{M}_* \otimes_\pi \langle\mathcal{E}\rangle \to \mathcal{M}_*$, $T(\omega \otimes a) = \omega a$, where $\langle\mathcal{E}\rangle$ denotes the closed linear span of $\mathcal{E}$. Consider the subset $Y_{\mathcal{E}} = \{\omega e \otimes f - \omega f \otimes e : e, f \in \mathcal{E}\} \subseteq \mathcal{M}_* \otimes_\pi \langle\mathcal{E}\rangle$, $Y = \langle Y_{\mathcal{E}}\rangle$ and put $Y_0 = Y \cap (\mathcal{M}_* \otimes \mathcal{E}_*)$, where $\mathcal{E}_*$ is the linear span of $\mathcal{E}$ in $\mathcal{M}$. Put $\mathcal{M}_* \widehat{\otimes}_{\mathcal{E}} \langle\mathcal{E}\rangle$ to be the quotient $(\mathcal{M}_* \otimes_\pi \langle\mathcal{E}\rangle)/Y$, whereas $\mathcal{M}_* \otimes_{\mathcal{E}} \mathcal{E}_*$ denotes the quotient normed space $(\mathcal{M}_* \otimes \mathcal{E}_*)/Y_0$. We prove that $\mathcal{M}_* \widehat{\otimes}_{\mathcal{E}} \langle\mathcal{E}\rangle = \mathcal{M}_*$ and $\mathcal{M}_* \otimes_{\mathcal{E}} \mathcal{E}_* = \mathcal{M}_{*\mathcal{E}}$ up to the isometric identifications given by T. Based on these equalities we prove that the bornological dual $(\mathcal{M}_{*\mathcal{E}})'$ of $\mathcal{M}_{*\mathcal{E}}$ is reduced to $\mathcal{M}_{\mathcal{E}}$ up to an isometry of the polynormed spaces, that is, $\mathcal{M}_{*\mathcal{E}}$ is the unique bornological predual of $\mathcal{M}_{\mathcal{E}}$.

In the abelian case, we have $\mathcal{M} = L^\infty(\mathcal{T})$ for a locally compact space $\mathcal{T}$ equipped with a positive Radon integral $\int : C_c(\mathcal{T}) \to \mathbb{C}$, the domain $\mathcal{E}$ in $\mathcal{M}$ is reduced to a measurable covering of $\mathcal{E}$, and the completion