J. Math. Anal. Appl. 466 (2018) 609–618

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Moduli $R(a, X)$ and $M(X)$ of direct sums of Banach spaces

Mariusz Szczepanik

Institute of Mathematics, Maria Curie-Skłodowska University, 20-031 Lublin, Poland

ARTICLE INFO

Article history:
Received 1 March 2018
Available online 15 June 2018
Submitted by T. Domínguez Benavides

Keywords:
Banach space
Direct sum
Fixed point

ABSTRACT

The moduli $R(a, X)$ and $M(X)$, introduced by Domínguez Benavides, play an important role in the fixed point theory for nonexpansive mappings. In the paper we show that if $\inf_{i\in I} M(X_i) > 1$, then $M\left(\left(\bigoplus_{i\in I} X_i\right)_Z\right) > 1$, where $\left(\bigoplus_{i\in I} X_i\right)_Z$ is the direct sum of Banach spaces X_i with respect to a Banach lattice Z, under some conditions for Z and I. Similar results are obtained for the modulus $R(a, X)$.

1. Introduction

In [11] Domínguez Benavides introduced the coefficient $M(X)$ of a Banach space X and proved that if $M(X) > 1$, then X has the weak fixed point property for nonexpansive mappings. The class of Banach spaces satisfying the condition $M(X) > 1$ contains all uniformly nonsquare spaces [12]. This result solved a long-standing problem if uniformly nonsquare spaces have the fixed point property for nonexpansive mappings.

One of many problems in the geometry of Banach spaces is to study conditions under which geometric properties are preserved under passing to direct sums. In 1943 Day showed (see [6]) that the direct sum of a family of Banach spaces $\{X_i\}$ with respect to a proper function space Z is uniformly convex if and only if Z is uniformly convex and the spaces X_i have a common modulus of convexity. For brief survey of the geometry of various direct sums of finitely many Banach spaces with respect to a variety of generalizations of uniform convexity and uniform smoothness we refer the reader to [9].

In 1968 Belluce, Kirk and Steiner (see [2]) showed that the direct sum of two Banach spaces with normal structure, endowed with the maximum norm, also has normal structure. Results concerning weak normal structure of Banach spaces are given in [10].

In [8] and [16] the authors considered finite direct sums of Banach spaces X_i with $M(X_i) > 1$ and in [14] with $R(X_i) < 2$. The more general case of direct sums of the form $\left(\bigoplus_{i\in I} X_i\right)_Z$, where $\{X_i\}$ is a family of

E-mail address: szczepan@hektor.umcs.lublin.pl.

https://doi.org/10.1016/j.jmaa.2018.06.009
0022-247X/

Banach spaces and the norm in a direct sum comes from a Banach lattice Z, was considered in [13], [3], [4] and [15].

The present paper deals with arbitrary sums of Banach spaces X_i satisfying the condition $\inf\{M(X_i) : i \in I\} > 1$ or $\sup\{R(a, X_i) : i \in I\} < 1 + a$. Following [7, page 5], we consider the direct sum $\left(\bigoplus_{i\in I} X_i\right)_Z$ of a family $\{X_i\}_{i\in I}$ of Banach spaces. We study relations between $R(a, X_i)$, $M(X_i)$ and $R\left(a, \left(\bigoplus_{i\in I} X_i\right)_Z\right)$, $M\left(\left(\bigoplus_{i\in I} X_i\right)_Z\right)$, respectively.

2. Basic definitions and lemmas

A Banach space X is said to be uniformly convex if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in B_X$ and $\|x - y\| \geq \varepsilon$, then $\|x + y\| \leq 2(1 - \delta)$. The modulus of uniform convexity of X is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : x, y \in B_X, \|x - y\| \geq \varepsilon\right\},$$

where $\varepsilon \in [0, 2]$. Clearly X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$. Given $a \geq 0$, we put

$$R(a, X) = \sup \liminf_{n\to\infty} \|x + x_n\|,$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$, and all weakly null sequences $(x_n) \subset B_X$ such that $\lim_{m,n\to\infty,\, n\neq m} \|x_n - x_m\| \leq 1$. Obviously

$$R(a, X) \leq 1 + a. \tag{2.1}$$

We define

$$M(X) = \sup\left\{\frac{1 + a}{R(a, X)} : a > 0\right\}.$$

Let Z be a Banach lattice. We say that Z is strictly monotone if $\|x\| < \|y\|$ provided that $0 \leq x \leq y$ and $x \neq y$. One of moduli of uniform monotonicity of Z is defined by

$$\sigma_Z(\varepsilon) = \inf\{\|x + y\| - 1 : x, y \geq 0,\ \|x\| \geq 1,\ \|y\| \geq \varepsilon\},$$

where $\varepsilon \in [0, \infty]$. We say that Z is uniformly monotone if $\sigma_Z(\varepsilon) > 0$ for every $\varepsilon > 0$. If the dimension of Z is finite, then uniform monotonicity of Z is equivalent to strict monotonicity of Z. Obviously $\delta_X(\varepsilon)$ and $\sigma_Z(\varepsilon)$ are nondecreasing functions and $\sigma_Z(\varepsilon)$ is continuous. Note that if Z is uniformly convex, then Z is uniformly monotone (see [1]). The Riesz angle of Z is defined by

$$\alpha(Z) = \sup\{\|x \vee y\| : x, y \in B_Z,\ x, y \geq 0\}.$$

In the sequel we will need the following lemmas.

Lemma 2.1. *Let X be a Banach space. If $x, y \in X \setminus \{0\}$, then*

$$\|x + y\| \leq (\|x\| \wedge \|y\|)\left(\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| - 1\right) + (\|x\| \vee \|y\|).$$