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Well-posedness and blow-up phenomena for an integrable three-component Camassa–Holm system ☆

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ABSTRACT

This paper studies the Cauchy problem for an integrable three-component Camassa–Holm system. We first establish the local well-posedness with initial condition in Besov spaces. Then we prove a blow-up criteria by arguing inductively with respect to the regularity index. Moreover, we derive a Riccati-type differential inequality by using the structure of equations, and also prove a new blow-up criteria with sufficient conditions on initial condition, whose proof is based on the conservative property of potential densities along the characteristic.

1. Introduction

In this paper we consider the Cauchy problem for the following integrable three-component Camassa–Holm system with cubic nonlinearity:

$$\begin{cases} m_{1t} + u_2 g m_{1x} - m_3(u_{2x} f - u_2 g) - m_1(3u_2 f - m_3 u_2) = 0, \\ m_{2t} + u_2 g m_{2x} + m_2(3u_{2x} g + m_3 u_2) = 0, \\ m_{3t} + u_2 g m_{3x} - m_3(2u_2 f + u_{2x} g - m_3 u_2) = 0, \\ m_1|_{t=0} = m_{10}, \quad m_2|_{t=0} = m_{20}, \quad m_3|_{t=0} = m_{30}, \end{cases} \tag{1.1}$$

where

$$m_i = u_i - u_{ixx}, \ i = 1, 2, 3, \quad f = u_3 - u_{1x}, \quad g = u_1 - u_{3x}.$$

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The system (1.1) was proposed by Li, Liu and Popowicz in [26], which is associated with the following 3×3 matrix spectral problem

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}_x = \begin{pmatrix}0 & 0 & 1\\ \lambda m_1 & 0 & \lambda m_3\\ 1 & \lambda m_2 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}, \tag{1.2}$$

where m_1, m_2, m_3 are three potentials and λ is a constant spectral parameter. Meanwhile, the authors pointed out that the system (1.1) can be rewritten as a bi-Hamiltonian system by

$$\begin{pmatrix}m_1\\ m_2\\ m_3\end{pmatrix}_t = \mathscr{L}_1\begin{pmatrix}\frac{\delta H_0}{\delta m_1}\\ \frac{\delta H_0}{\delta m_2}\\ \frac{\delta H_0}{\delta m_3}\end{pmatrix} = \mathscr{L}_2\begin{pmatrix}\frac{\delta H_1}{\delta m_1}\\ \frac{\delta H_1}{\delta m_2}\\ \frac{\delta H_1}{\delta m_3}\end{pmatrix} \tag{1.3}$$

with

$$H_0 = \int (u_1 f + u_{2x} g) g m_2 dx, \quad H_1 = \int g m_2 dx,$$

where $\mathscr{L}_1$ and $\mathscr{L}_2$ denote the Hamiltonian operators in terms of variables m_1, m_2, m_3, see [26] for more details. Moreover, they have suggested the way to construct infinitely many conserved quantities for the integrable system (1.1).

If $u_3 \equiv 0$, the system (1.1) reduces to the two-component Novikov system

$$\begin{cases} m_t + uvm_x + 3vu_x m = 0,\\ n_t + uvn_x + 3uv_x n = 0,\end{cases} \tag{1.4}$$

where $m = u - u_{xx}$, $n = v - v_{xx}$. The system (1.4) was proposed by Geng and Xue [12], in which they calculated the N-peakons and conserved quantities and found a Hamiltonian structure. The associated bi-Hamiltonian structure for (1.4) was presented in [24]. It is proved that the Geng–Xue system also admits shockpeakons [28], which was first introduced for the Degasperis–Procesi equation [27]. Notice that if $u = v$ in (1.4), we obtain the famous Novikov equation [31]

$$m_t + 3uu_x m + u^2 m_x = 0, \quad m = u - u_{xx}, \tag{1.5}$$

which possesses a bi-Hamiltonian structure, an infinite sequence of conserved quantities and the explicit formulas for multipeakon solutions [20,21]. In the years immediately following, many attempts have been made to the studying of well-posedness, blow-up phenomena and analyticity for (1.5) in Sobolev spaces and Besov spaces, see for example [14,19,22,30,38].

By taking $u_2 \equiv 1$ and $u_3 \equiv 0$ in (1.1), it becomes the Degasperis–Procesi (DP) equation

$$m_t + um_x + 3u_x m = 0, \quad m = u - u_{xx}. \tag{1.6}$$

The formal integrability of DP equation can be proved by constructing a Lax pair, while the direct and inverse scattering approach to pursue it can be found in [8]. The DP equation has bi-Hamiltonian structure [8] and an infinite sequence of conserved quantities, and admits exact peakon solutions which are analogous to the Camassa–Holm peakons [2,6,7]. It is worth pointing out that solutions of this type are not mere abstractizations: the peakons replicate a feature that is characteristic for the waves of great height—waves of largest amplitude that are exact solutions of the governing equations for irrotational water waves [3,33]. Recently, by applying the dressing method to smooth localized solutions of (1.6), Constantin and Ivanov