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Closure properties for integral problems driven by regulated functions via convergence results

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A B S T R A C T

In this paper we give necessary and sufficient conditions for the convergence of Kurzweil–Stieltjes integrals with respect to regulated functions, using the notion of asymptotical equiintegrability. One thus generalizes several well-known convergence theorems. As applications, we provide existence and closure results for integral problems driven by regulated functions, both in single- and set-valued cases. In the particular setting of bounded variation functions driving the equations, we get features of the solution set of measure integrals problems.

1. Introduction

The role of convergence results for integrals in the theory of differential and integral equations is well-known. On the other hand, when studying a large number of problems one can notice the appearance of discontinuities in the behaviour of the functions, so we are led to the idea of working with measure driven problems, i.e.

$$x(t) = x_0 + \int_0^t f(s, x(s))dg(s) \tag{1}$$

or its multivalued counterpart,

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$$x(t) \in x_0 + \int_0^t F(s, x(s))dg(s), \tag{2}$$

where X is a Banach space, $\mathcal{P}_{cc}(X)$ is the family of all nonempty, closed and convex subsets of X, g is a real bounded variation function, $x_0 \in X$ and $f : [0,1] \times X \to X$, $F : [0,1] \times X \to \mathcal{P}_{cc}(X)$ are functions, resp. multifunctions.

There is a wide literature treating this subject (we refer to [1], [9], [13], [14] in the single-valued case and to [8], [11], [31], [37] in the set-valued setting). The motivation comes from the fact that one can thus cover the framework of usual differential problems (when g is absolutely continuous), of discrete problems (when g is a sum of step functions), of impulsive equations (for g being the sum between an absolutely continuous function and a sum of step functions), as well as retarded problems (see [1]). As proven in [13], dynamic equations on time scales and generalized differential equations can also be seen as measure differential equations.

On the other hand, it is of interest to develop an existence theory for this kind of problems in the more permissive case where the function g is only regulated (i.e. it has one-sided limits at every point) but it is not an easy task since the properties of primitives with respect to such functions are very weak (see e.g. [20] or [38]).

It is also important to have closure results for the studied problem, namely to check if when considering a sequence $(g_n)_n$ of functions converging to a function g the solutions of the equation governed by g_n is "close" (in some sense to be specified) to solutions of the equation governed by g.

To this purpose, it is necessary to have a convergence result for Stieltjes integrals of the following form:

$$\lim_{n\to\infty} \int_0^1 f_n(s)dg_n(s) = \int_0^1 f(s)dg(s)$$

and since when working with regulated functions the most appropriate integration theory is the Kurzweil–Stieltjes one, we focus in the first section of our paper (after the Preliminaries) on the matter of proving such a convergence theorem for the Kurzweil–Stieltjes integral.

Thus, we prove a necessary and sufficient assertion: the convergence holds if and only if f_n is asymptotically equiintegrable w.r.t. g_n on the unit interval. This is a concept (introduced in [2]) which encompasses that of equiintegrability, often implied when looking for the convergence of integrals. Our result generalizes [2, Theorem 8.12] where the functions are real-valued and $g_n = g$ for every $n \in \mathbb{N}$ and it is more general when compared to other results of convergence type (see Section 3).

Next, we apply the main theorem to get the existence of regulated solutions for integral equations and inclusions driven by regulated functions in general Banach spaces. In the single-valued case we apply a version of Schauder's fixed point theorem, while in the multivalued case we make use of a nonlinear alternative of Leray–Schauder type. In both situations, one of the main tools is the notion of equiregulatedness of a set of regulated functions (see [15]).

Afterwards, we focus on the closure properties of the solutions set for such problems; namely, to study if, when taking a sequence of regulated functions $(g_n)_n$ converging to a regulated function g, the solution set of the problem governed by g_n is close (in a specified sense) to the solution set of the problem governed by g. Such results are obtained via our main convergence theorem and are very important (in numerical analysis, for instance) since they allow one to study a general integral problem governed by a rough function by analysing similar problems governed by functions with much better properties.

We relate them to well-known results in literature in the case of problems governed by functions of bounded variation ([9], [18], [13], [25] in the single-valued case or [37], [31] in the set-valued setting).