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A q-analogue of the (J.2) supercongruence of Van Hamme

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ABSTRACT

Zudilin proved some Ramanunjan-type supercongruences by the WZ method. Long proved a Ramanujan-type supercongruence conjecture due to Van Hamme by applying some hypergeometric evaluation identities. In this paper, we propose a conjecture on a complete q-analogue of this supercongruence of Van Hamme and prove it in a weaker form via the q-WZ method. Additionally, we give a q-analogue of a related congruence involving cubes of binomial coefficients due to Sun in the same way. We also present a conjecture on a q-analogue of the corresponding infinite series for $1/\pi$ due to Ramanujan.

1. Introduction

In his second notebook, Ramanujan recorded the following formula for $1/\pi$ (see [2, p. 352]):

$$\sum_{k=0}^{\infty} \frac{(6k+1)(\frac{1}{2})_k^3}{k!^3 4^k} = \frac{4}{\pi}, \tag{1.1}$$

where we use the Pochhammer symbol $(a)_k = a(a+1)\cdots(a+k-1)$. The identity (1.1) can also be found in Ramanujan's paper [22] together with some other similar examples that enable us to compute π very accurately.

Curiously, a proof of (1.1) was not found until 1987, when J.M. Borwein and P.B. Borwein proved it in their book [4, pp. 177–187]. In 1997, Van Hamme conjectured a p-adic analogue of (1.1) as follows:

Entry (J.2) (Van Hamme [29]). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{(6k+1)(\frac{1}{2})_k^3}{k!^3 4^k} \equiv (-1)^{\frac{p-1}{2}} p \pmod{p^4}. \tag{1.2}$$

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Supercongruences of this type are called Ramanujan-type supercongruences. Entry (J.2) is one of 13 Ramanujan-type supercongruences originally conjectured by Van Hamme [29]. All of the 13 supercongruences have now been confirmed using a variety of techniques (see [20,27] for historic remarks on this). For example, the supercongruence (B.2) was first proved by Mortenson [18] using a ${}_6F_5$ transformation and a technical evaluation of a quotient of Gamma functions. The entry (J.2) was proved by Long [16] using hypergeometric identities. Zudilin [33] adopted the method of Wilf–Zeilberger (WZ) pairs not only to give another proof of (B.2), but also to demonstrate several new Ramanujan-type supercongruences. Nevertheless, Zudilin [33] pointed out that the known WZ pairs can only be used to prove the supercongruence (J.2) modulo p^2.

On the other hand, the author and Zeng [13, Corollary 1.2] have given a q-analogue of (H.2). Motivated by Zudilin's work [33], the author [10,11] used the q-WZ method to obtain q-analogues of (B.2), (E.2), and (F.2). The author and Wang [12] used a variation of the q-WZ method to prove a q-analogue of [16, Theorem 1.1]. Thus a q-analogue of (C.2) is also known. Note that some other interesting q-congruences were given in [15,21,23,24,28].

In this paper we shall give a complete q-analogue of (J.2). Recall that the *q-shifted factorial* is defined by $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 1$ and $(a;q)_0 = 1$, while the *q-integer* is defined as $[n] = [n]_q = (1-q^n)/(1-q)$. The n-th *cyclotomic polynomial* $\Phi_n(q)$ is given by

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}} (q - e^{2\pi i \frac{k}{n}}),$$

where i is the imaginary unit. It is clear that $\Phi_p(q) = [p]$ for any prime p. Some other basic properties of cyclotomic polynomials can be found in [19].

Our q-analogue of Van Hamme's supercongruence (J.2) can be stated as follows:

Conjecture 1.1. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{\frac{n-1}{2}} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} \equiv [n](-q)^{\frac{1-n}{2}} + \frac{(n^2-1)(1-q)^2}{24}[n]^3(-q)^{\frac{1-n}{2}} \pmod{[n]\Phi_n(q)^3}. \tag{1.3}$$

Clearly, the congruence (1.3) modulo $[n]\Phi_n(q)^2$ reduces to

$$\sum_{k=0}^{\frac{n-1}{2}} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} \equiv [n](-q)^{\frac{1-n}{2}} \pmod{[n]\Phi_n(q)^2}. \tag{1.4}$$

It is interesting that (1.4) has an accompanying congruence as follows:

Conjecture 1.2. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{n-1} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} \equiv [n](-q)^{\frac{1-n}{2}} \pmod{[n]\Phi_n(q)^2}. \tag{1.5}$$

Note that, when $n = p$ is an odd prime, the congruences (1.4) and (1.5) modulo $[p]^3$ are equivalent to each other, since $\frac{(q;q^2)_k}{(q^4;q^4)_k} \equiv 0 \pmod{[p]}$ for $(p+1)/2 \leqslant k \leqslant p-1$. But for general n they are clearly different congruences.

The first purpose of this paper is to prove the following weaker form of Conjectures 1.1 and 1.2.