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Some evaluation of cubic Euler sums

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ABSTRACT

P. Flajolet and B. Salvy [15] prove the famous theorem that a nonlinear Euler sum $S_{i_1 i_2 \cdots i_r, q}$ reduces to a combination of sums of lower orders whenever the weight $i_1 + i_2 + \cdots + i_r + q$ and the order r are of the same parity. In this article, we develop an approach to evaluate the cubic sums $S_{1^2 m,p}$ and $S_{1 l_1 l_2, l_3}$. By using the approach, we establish some relations involving cubic, quadratic and linear Euler sums. Specially, we prove the cubic sums $S_{1^2 m,m}$ and $S_{1(2l+1)^2,2l+1}$ are reducible to zeta values, quadratic and linear sums. Moreover, we prove that the two combined sums involving multiple zeta values of depth four

$$\sum_{\{i,j\}\in\{1,2\}, i\neq j} \zeta(m_i, m_j, 1, 1) \quad \text{and} \quad \sum_{\{i,j,k\}\in\{1,2,3\}, i\neq j\neq k} \zeta(m_i, m_j, m_k, 1)$$

can be expressed in terms of multiple zeta values of depth ≤ 3, here $2 \leq m_1, m_2, m_3 \in \mathbb{N}$. Finally, we evaluate the alternating cubic Euler sums $S_{\bar{1}^3, 2r+1}$ and show that it are reducible to alternating quadratic and linear Euler sums. The approach is based on Tornheim type series computations.

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1. Introduction

In response to a letter from Goldbach, Euler considered sums of the form (see Berndt [4] for a discussion)

$$S_{p,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}, \tag{1.1}$$

and was able to give explicit values for certain of these sums in terms of the Riemann zeta function, where p, q are positive integers with $q \geq 2$, and $w := p + q$ denotes the weight of linear sums $S_{p,q}$. These kinds of sums are called the linear Euler sums (for short linear sums) today. Here $H_n^{(p)}$ denotes the harmonic number which is defined by

$$H_n^{(k)} := \sum_{j=1}^{n} \frac{1}{j^k} \quad \text{and} \quad H_0^{(k)} := 0.$$

When $k = 1$, then $H_n := H_n^{(1)}$, which is called the classical harmonic number.

In their famous paper [15], Flajolet and Salvy introduced the following generalized series

$$S_{\mathbf{S},q} := \sum_{n=1}^{\infty} \frac{H_n^{(s_1)} H_n^{(s_2)} \cdots H_n^{(s_r)}}{n^q}, \tag{1.2}$$

which is called the generalized (nonlinear) Euler sums. Here $\mathbf{S} := (s_1, s_2, \ldots, s_r)$ $(r, s_i \in \mathbb{N}, i = 1, 2, \ldots, r)$ and $q \geq 2$. The quantity $w := s_1 + \cdots + s_r + q$ is called the weight and the quantity r is called the degree. As usual, repeated summands in partitions are indicated by powers, so that for instance

$$S_{1^2 2^3 4,q} = S_{112224,q} = \sum_{n=1}^{\infty} \frac{H_n^2 [H_n^{(2)}]^3 H_n^{(4)}}{n^q}.$$

It has been discovered in the course of the years that many Euler sums admit expressions involving finitely the "zeta values", that is to say of the Riemann zeta function [2],

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \Re(s) > 1$$

with positive integer arguments. For example, the linear sums $S_{p,q}$ can be evaluated in terms of zeta values in the following cases: $p = 1, p = q, p + q$ odd and $p + q = 6$ with $q \geq 2$ (for more details, see [3,5,15]). In 1994, Bailey et al. [3] proved that all Euler sums of the form $S_{1^p,q}$ for weights $p + q \in \{3, 4, 5, 6, 7, 9\}$ are reducible to $\mathbf{Q}$-linear combinations of zeta values by using the experimental method. In [26,30], we proved that all Euler sums of weight ≤ 7 are reducible to $\mathbb{Q}$-linear combinations of single zeta monomials. For weight 9, all Euler sums of the form $S_{s_1 \cdots s_k,q}$ with $q \in \{4, 5, 6, 7\}$ are expressible polynomially in terms of zeta values. Very recently, Wang et al. [25] shown that all Euler sums of weight nine are reducible to zeta values.

However, there are also many nonlinear Euler sums which need not only zeta values but also linear sums. Namely, many nonlinear Euler sums are reducible to polynomials in zeta values and to linear sums (see [3,5,15,19,29,31]). For instance, in 1995, Borwein et al. [5] showed that the quadratic sums $S_{1^2,q}$ can reduce to linear sums $S_{2,q}$ and polynomials in zeta values. In 1998, Flajolet and Salvy [15] used the contour integral representations and residue computation to show that the quadratic sums $S_{p_1 p_2,q}$ are reducible to linear sums and zeta values when the weight $p_1 + p_2 + q$ is even and $p_1, p_2 > 1$. In [26], we proved that all Euler sums with weight eight are reducible to zeta values and linear sum $S_{2,6}$.