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# Bohr's inequalities for the analytic functions with lacunary series and harmonic functions

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ABSTRACT

We determine the Bohr radius for the class of all functions $f$ of the form $f(z) = z^m \sum_{k=0}^{\infty} a_{kp} z^{kp}$ analytic in the unit disk $|z| < 1$ and satisfy the condition $|f(z)| \leq 1$ for all $|z| < 1$. In particular, our result also contains a solution to a recent conjecture of Ali et al. [9] for the Bohr radius for odd analytic functions, solved by the authors in [17]. We consider a more flexible approach by introducing the $p$-Bohr radius for harmonic functions which in turn contains the classical Bohr radius as special case. Also, we prove several other new results and discuss $p$-Bohr radius for the class of odd harmonic bounded functions.



## 1. Preliminaries

In 1914, H. Bohr [13] proved that if the power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ converges in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $|f(z)| < 1$ for all $z \in \mathbb{D}$, then the majorant series $M_f(r) := \sum_{k=0}^{\infty} |a_k| r^k$ is less than or equal to 1 for all $|z| = r \leq 1/6$. The largest $r \leq 1$ such that the above inequality holds is referred to as the Bohr radius for the unit disk case. The fact that the constant $1/3$ is best possible was established independently by F. Wiener, M. Riesz and I. Schur. Other proofs of this result were later obtained by Sidon and Tomic. Bohr's idea naturally extends to functions of several complex variables and thus, a variety of results on Bohr's theorem in higher dimension appeared recently. In this contexts and in other respects, we suggest the reader to refer [1–7,10–12,18,19] and the references there. For a detailed account of the development, we refer to the recent survey article on this topic [8] and the references therein. More recently, the present authors obtained the following result as a corollary to a general result for symmetric functions and thereby settling the recent conjecture of Ali et al. [9].

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**Theorem A.** ([17, Corollary 1]) *If $f(z) = z\sum_{k=0}^{\infty} a_{2k}z^{2k}$ is odd analytic function in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$, then $M_f(r) \leq 1$ for $r \leq r_2 = 0.789991\ldots$. The extremal function has the form $z(z^2 - a)/(1 - az^2)$.*

Motivated by the work of Ali et al. [9] and Theorem A, we raise the following

**Problem 1.** ([9]) Given $p \in \mathbb{N}$ and $0 \leq m \leq p$, determine the Bohr radius for the class of functions $f(z) = z^m \sum_{k=0}^{\infty} a_{pk}z^{pk}$ analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$.

The case $m = 1$ has been handled by the authors in [17].

One of the aims of this article is to solve this problem completely and present several of its consequences. As remarked in [17], it brings serious difficulties because if we use sharp the inequalities $|a_n| \leq 1 - |a_0|^2$ ($n \geq 1$) simultaneously (as in the classical case) then we will not be able to obtain sharp result due to the fact that in the extremal case $|a_0| < 1$. Also it is important that in the classical case there is no extremal function while in our case there is.

The paper is organized as follows. The solution to Problem 1 is presented in Section 2 (see Theorem 1) and its proof is given in Section 3. In Section 4, we introduce the concept of $p$-Bohr radius for the class of bounded harmonic functions defined on the unit disk and discuss the Bohr inequality for planar harmonic functions as a special case. We expect that our approach will lead to several new investigations on the general notion of the so-called Bohr's phenomenon.

## 2. Bohr inequality for analytic functions of the form $z^m \sum_k b_k z^{kp}$

**Theorem 1.** *Let $p \in \mathbb{N}$ and $0 \leq m \leq p$, $f(z) = z^m \sum_{k=0}^{\infty} a_{pk}z^{pk}$ be analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$. Then*

$$M_f(r) \leq 1 \quad \text{for} \quad r \leq r_{p,m},$$

*where $r_{p,m}$ is the maximal positive root of the equation*

$$-6r^{p-m} + r^{2(p-m)} + 8r^{2p} + 1 = 0. \tag{1}$$

*The number $r_{p,m}$ is sharp. Moreover, in the case $m \geq 1$ there exists an extremal function of the form $z^m(z^p - a)/(1 - az^p)$, where*

$$a = \left(1 - \frac{\sqrt{1 - {r_{p,m}}^{2p}}}{\sqrt{2}}\right)\frac{1}{{r_{p,m}}^p}.$$

The case $p = 2$ and $m = 1$ has a special interest which is indeed Theorem A and it provides a solution to the conjecture of Ali et al. [9]. More generally, it is a simple exercise to see that for the case $p = m, 2m, 3m$, the Bohr radii give

$$r_{m,m} = 1/\sqrt[2m]{2},\ r_{2m,m} = \sqrt[m]{r_2}, \ \text{ and } \ r_{3m,m} = \sqrt[2m]{\frac{7+\sqrt{17}}{16}},$$

respectively, where $r_2$ is given in Theorem A. It is worth pointing out from the last case that $r_{3,1}$ gives the value $(\sqrt{7+\sqrt{17}})/4$. The result for $m = 0$ is well known [9] which we now recall because of its independent interest.

**Corollary 1.** *Let $p \geq 1$. If $f(z) = \sum_{k=0}^{\infty} a_{pk}z^{pk}$ is analytic in $\mathbb{D}$, and $|f(z)| \leq 1$ in $\mathbb{D}$, then $M_f(r) \leq 1$ for $0 \leq r \leq r_{p,0} = 1/\sqrt[p]{3}$. The radius $r_{p,0} = 1/\sqrt[p]{3}$ is best possible.*