J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Smallness and cancellation in some elliptic systems with measure data

Francesco Leonetti [a], Eugénio Rocha [b], Vasile Staicu [b]

[a] Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università di L'Aquila, 67100 L'Aquila, Italy
[b] CIDMA – Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

A R T I C L E I N F O

Article history:
Received 28 February 2018
Available online xxxx
Submitted by A. Cianchi

Keywords:
Elliptic
System
Existence
Solution
Measure

A B S T R A C T

In a bounded open subset $\Omega \subset \mathbb{R}^n$, we study Dirichlet problems with elliptic systems, involving a finite Radon measure μ on $\mathbb{R}^n$ with values into $\mathbb{R}^N$, defined by

$$\begin{cases} -\mathrm{div}\, A(x, u(x), Du(x)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $A_i^\alpha(x, y, \xi) = \sum_{\beta=1}^{N} \sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x, y)\, \xi_j^\beta$ with $\alpha \in \{1, \ldots, N\}$ the equation index. We prove the existence of a (distributional) solution $u : \Omega \to \mathbb{R}^N$, obtained as the limit of approximations, by assuming: (i) that coefficients $a_{i,j}^{\alpha,\beta}$ are bounded Carathéodory functions; (ii) ellipticity of the diagonal coefficients $a_{i,j}^{\alpha,\alpha}$; and (iii) smallness of the quadratic form associated to the off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$ (i.e. $\alpha \neq \beta$) verifying a r-staircase support condition with $r > 0$. Such a smallness condition is satisfied, for instance, in each one of these cases: (a) $a_{i,j}^{\alpha,\beta} = -a_{j,i}^{\beta,\alpha}$ (skew-symmetry); (b) $|a_{i,j}^{\alpha,\beta}|$ is small; (c) $a_{i,j}^{\alpha,\beta}$ may be decomposed into two parts, the first enjoying skew-symmetry and the second being small in absolute value. We give an example that satisfies our hypotheses but does not satisfy assumptions introduced in previous works. A Brezis's type nonexistence result is also given for general (smooth) elliptic-hyperbolic systems.

1. Introduction

Let us consider the Dirichlet elliptic problem

$$-\mathrm{div}\, [A(x, u(x), Du(x))] = \mu \quad \text{in } \Omega, \tag{1.1}$$

E-mail addresses: leonetti@univaq.it (F. Leonetti), eugenio@ua.pt (E. Rocha), vasile@ua.pt (V. Staicu).

https://doi.org/10.1016/j.jmaa.2018.05.047
0022-247X/

Please cite this article in press as: F. Leonetti et al., Smallness and cancellation in some elliptic systems with measure data, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.05.047

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, μ is a measure on $\mathbb{R}^n$ with values into $\mathbb{R}^N$ and A satisfies suitable coercivity and growth conditions. We note that (1.1) is a system of N equations.

First consider the case $N = 1$, i.e. (1.1) is only one single equation. Existence of distributional solutions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ has been deeply studied, starting from [11], see [16], [25], [12], [61], [7] and the survey [8]. Uniqueness seems to be a delicate matter, e.g. see [62], [5], [32] and the introduction of [24]. Regularity results are contained in [56], [57], [59], [17], [18], [58], [35], [40], [3], [2], [14] and the survey [60] (see also [9] and [10]). Note that existence of solutions is usually obtained by a truncation argument, which shows why the vectorial case $N \geq 2$ is difficult and only few contributions are available in the literature. In fact, for systems $N \geq 2$, the p-Laplacian $A(x, y, \xi) = |\xi|^{p-2}\xi$ is treated in [31] and [26], and the anisotropic case, in which each component of the gradient $D_i u$ may have a possibly different exponent p_i, is dealt in [50] and [51]. Let us mention [41] for pointwise potential estimates in the framework of vectorial p-Laplacian. Let us write (1.1) using components, that is,

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[A_i^\alpha(x, u(x), Du(x))\right] = \mu^\alpha \quad \text{for } \alpha \in \{1, \ldots, N\}. \tag{1.3}$$

We note that systems more general than the p-Laplacian are considered in [27] and [29], under the assumption

$$0 \leq \sum_{\alpha=1}^{N} \sum_{i=1}^{n} A_i^\alpha(x, y, \xi)((Id - b \times b)\xi)_i^\alpha \tag{1.4}$$

for every $b \in \mathbb{R}^N$ with $|b| \leq 1$; see also [42], [6], [20] where (1.4) has been used. In [63], the author assumes the componentwise sign condition

$$0 \leq \sum_{i=1}^{n} A_i^\alpha(x, y, \xi)\xi_i^\alpha \tag{1.5}$$

for every $\alpha \in \{1, ..., N\}$. When $N = 2$, (1.4) implies (1.5), since it is enough to take first $b = (1, 0)$ and then $b = (0, 1)$. In [52], the authors consider that A is independent of y and satisfies the componentwise coercivity condition

$$\nu|\xi^\alpha|^2 - M \leq \sum_{i=1}^{n} A_i^\alpha(x, \xi)\xi_i^\alpha \tag{1.6}$$

for every $\alpha \in \{1, ..., N\}$, for some constants $\nu \in (0, +\infty)$ and $M \in [0, +\infty)$. In [28], they relax (1.4) to some extent

$$-c|\xi|^q - g(x) \leq \sum_{\alpha=1}^{N} \sum_{i=1}^{n} A_i^\alpha(x, y, \xi)((Id - b \times b)\xi)_i^\alpha \tag{1.7}$$

for some $c \in [0, +\infty)$, $g \in L^1(\Omega)$ and $q \in [1, n)$ where $\xi \mapsto A(x, y, \xi)$ is n-coercive.

We note that in [32], [30] and [53] the authors do not truncate u. They modify Du and then adjust via Hodge decomposition; such a procedure requires the dimension n to be the exponent in the coercivity condition for A. The authors use nice estimates for Hodge decomposition, which have been studied in [33] (see also appendix A, in [34]).

Please cite this article in press as: F. Leonetti et al., Smallness and cancellation in some elliptic systems with measure data, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.05.047