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An algebraic approach to tempered ultradistributions ☆

Smiljana Jakšić [a], Snježana Maksimović [b], Stevan Pilipović [c,*],

[a] *Faculty of Forestry, University of Belgrade, Kneza Višeslava 1, 11000 Belgrade, Serbia*
[b] *Faculty of Electrical Engineering, University of Banja Luka, Patre 5, 78000 Banja Luka, Bosnia and Herzegovina*
[c] *Faculty of Sciences, University of Novi Sad, Trg D. Obradovica 4, 21000 Novi Sad, Serbia*

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A B S T R A C T

We construct the space of pseudo-quotients that is shown to be isomorphic to the spaces of Beurling tempered ultradistributions.

1. Introduction

In our study, we were motivated by the results from [1] where an isomorphism between the space of tempered distributions and the space of pseudo-quotients including their convergence structures is presented. The main goal of our paper is to obtain such a result for the spaces of Beurling tempered ultradistributions. Our approach here is quite different than one in [1] which utilize the framework of pseudo-quotients. Rather it is based on an intrinsic analysis of a class of ultradifferential operators and corresponding structural theorems.

An open problem that still needs to be solved is to build a space of pseudo-quotients that is isomorphic to the space of Roumieu tempered ultradistributions.

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* Corresponding author.
E-mail addresses: smiljana.jaksic@sfb.bg.ac.rs (S. Jakšić), snjezana.maksimovic@etf.unibl.org (S. Maksimović), pilipovic@dmi.uns.ac.rs (S. Pilipović).

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1.1. Notation

We employ the notation $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ for the sets of positive integers, real and complex numbers, respectively; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given a multi-index $n \in \mathbb{N}_0^d$, for $x \in \mathbb{R}^d$ we write as usual $x^n = x_1^{n_1} \cdot ... \cdot x_d^{n_d}$, $D^n = D_x^n = D_1^{n_1} \cdots D_d^{n_d}$, where $D_k = \frac{1}{i}\frac{\partial}{\partial x_k}$, $k = 1, ..., d$, and for $\alpha, \beta \in \mathbb{N}_0^d$, $\alpha = (\alpha_1, ..., \alpha_d)$, $\beta = (\beta_1, ..., \beta_d)$, $\binom{\alpha}{\beta} = \prod_{k=1}^d \binom{\alpha_k}{\beta_k}$, $\beta \leq \alpha$ means $\beta_k \leq \alpha_k$, $k = 1, ..., d$; $|\alpha| = \alpha_1 + ... + \alpha_d$. We fix the constants in the Fourier transform as $\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{i\langle x,\xi\rangle} dx$.

1.2. Ultradifferentiable functions of ultrapolynomial growth

Following the approach from [3] (cf. [2]), we introduce the test spaces for the spaces of tempered ultradistributions through the use of positive sequences $(M_p)_{p\in\mathbb{N}_0}$ of real numbers which satisfy the following conditions:

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$, $p \in \mathbb{N}$;
$(M.2)$ There are constants A, $H > 1$ such that $M_p \leq AH^p \min_{0\leq q\leq p} M_q M_{p-q}$, $p, q \in \mathbb{N}_0$;
$(M.3)$ There is a constant A such that $\sum_{p=q+1}^{\infty} M_{p-1}/M_p < Aq\, M_q/M_{q+1}$, $q \in \mathbb{N}$;

We always assume $M_0 = 1$. The associated function for the sequence $(M_p)_{p\in\mathbb{N}}$ is defined as

$$M(\lambda) = \sup_{p\in\mathbb{N}} \log \frac{\lambda^p}{M_p},\ \lambda > 0 \text{ and } M(0) = 0.$$

We refer to [3] for the meaning of these three conditions and their translation into properties of M. In particular, the condition $(M.2)$ is equivalent to

$$2M(\lambda) \leq M(H\lambda) + \ln A,\ \lambda > 0 \text{ (cf. [3, Proposition 3.6, p. 50]).} \tag{1.1}$$

Let $m_p = M_p/M_{p-1},\ p \in \mathbb{N}$. We denote by $m(\rho)$ the number of $m_p \leq \rho$. Then, we have

$$M(\lambda) = \int_0^{\lambda} \frac{m(\rho)}{\rho} d\rho,\ \lambda > 0 \text{ (cf. [3], (3.11), p. 50).} \tag{1.2}$$

For the sake of simplicity, we consider $M_p = p!^s$ with $s > 1$. In this case $m_p = p^s$. Hence, it follows from (1.2)

$$M(\lambda) \asymp s\lambda^{1/s}, \lambda > 0,$$

that is $M(\lambda)/(s\lambda^{1/s}) \to 1$, as $\lambda \to \infty$. Moreover, for every $\varepsilon \in (0, 1)$ there exists λ_0 such that $(1-\varepsilon)s\lambda^{1/s} \leq M(\lambda) \leq s\lambda^{1/s}, \lambda > \lambda_0$.

Let $h > 0$. By $\mathcal{S}_h^{(s)}(\mathbb{R}^d)$ we denote the Banach space of all smooth functions φ on $\mathbb{R}^d$ such that the norm

$$\sigma_h(\varphi) = \sum_{n,k\in\mathbb{N}_0^d} \frac{h^{|n+k|}}{n!^s k!^s} \|x^n D^k \varphi\|_{L^2}$$

is finite. We define the space $\mathcal{S}^{(s)}(\mathbb{R}^d)$ as a projective limit of the spaces $\mathcal{S}_h^{(s)}(\mathbb{R}^d)$

$$\mathcal{S}^{(s)}(\mathbb{R}^d) = \varprojlim_{h\to\infty} \mathcal{S}_h^{s}(\mathbb{R}^d).$$