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# Sharp Hardy constants for annuli

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ABSTRACT

We consider the Hardy inequality in canonical doubly connected plane domains. For any annulus $A$ we determine sharp Hardy's constant $c_2(A)$ in function of conformal modulus $M(A)$. Namely, for any annulus $A$ with fixed conformal modulus $M(A) = M$ we prove that

$$c_2(A) = \begin{cases} 1/4, & \text{if } M \in (0, M^*]; \\ \gamma(2-\gamma)/4, & \text{if } M \in (M^*, \infty), \end{cases}$$

where $\gamma = \gamma(M) \in (1, 2)$. The critical modulus $M^* \approx 0.57298$ and the values of $\gamma(M)$ are found as roots of certain equations, containing the Gauss hypergeometric functions. In particular, we show that the sharp Hardy constants $c_2(A)$ depend on $M$ continuously and that they tend to zero as $M \to \infty$. In addition, we describe an application of results to a Rellich type inequality.



## 1. Introduction

Let $\Omega \subset \mathbb{C}$ be a plane domain such that $\Omega \neq \mathbb{C}$. We consider functions $\varphi : \Omega \to \mathbb{R}$ and the following Hardy inequality

$$\iint_\Omega |\nabla\varphi(z)|^2 dxdy \geq c_2(\Omega) \iint_\Omega \frac{\varphi^2(z)}{(\text{dist}(z, \partial\Omega))^2} dxdy, \quad \forall \varphi \in C_0^\infty(\Omega), \tag{1}$$

where $z = x + iy$, $\text{dist}(z, \partial\Omega) := \inf_{\zeta\in\partial\Omega} |z - \zeta|$ is the distance from a point $z \in \Omega$ to the boundary of the domain. We suppose that the quantity $c_2(\Omega)$ is defined as the best possible constant, i.e.

$$c_2(\Omega) = \inf_{\varphi\in C_0^\infty(\Omega), \varphi\not\equiv 0} \frac{\iint_\Omega |\nabla\varphi(z)|^2 dxdy}{\iint_\Omega \varphi^2(z)(\text{dist}(z, \partial\Omega))^{-2} dxdy}. \tag{2}$$

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There are many results connected with inequality (1) (see, for instance, [2], [7], [9], [11], [12] and the bibliography in [9]). In particular, it is well known that $c_2(\Omega) > 0$, i.e. inequality (1) is non-trivial, for any bounded domain $\Omega$ with locally Lipschitz boundary. From (2) it follows that $c_2(\Omega) = c_2(a\Omega + b)$, where $a, b \in \mathbb{C}$, $a \neq 0$, and $a\Omega + b = \{az + b : z \in \Omega\}$.

The sharp value of $c_2(\Omega)$ is known for convex domains. Namely, several authors independently proved that $c_2(\Omega) = 1/4$ for any convex domain $\Omega \neq \mathbb{C}$ (see a detailed description in [3], [4], [9], [10]). Also, it is proved that $c_2(\Omega) = 1/4$ for some non-convex domains close to convex in a certain sense (see [4], [9], [10]). By the way, the following Davies problem [10] is still open: is it true that $c_2(\Omega) \leq 1/4$ for any plane domain $\Omega \neq \mathbb{C}$?

The aim of this paper is to find sharp Hardy constants for annuli of the form

$$A = A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\},\ 0 < r_1 < r_2 < \infty.$$

Since $c_2(A) = c_2(aA + b)$, $a \neq 0$, the constant $c_2(A(z_0, r_1, r_2))$ depends on the conformal modulus

$$M(A(z_0, r_1, r_2)) := \frac{1}{2\pi} \ln \frac{r_2}{r_1},$$

only. In addition, one can find the following properties of $c_2(A)$.

(i) From [10] it follows that $c_2(A) \leq 1/4$ for any annulus $A$ with finite modulus. (ii) It is known [3] that $c_2(A) = O(M^{-2}(A))$ as $M(A) \to \infty$. In particular, $c_2(A(z_0, 0, 1)) = 0$. (iii) From Theorem 1 in [4] it follows that $c_2(A) = 1/4$ for annuli with sufficiently small moduli. More precisely, it is proved that $c_2(A) = 1/4$ for every annulus $A = A(z_0, r_1, r_2)$ satisfying the condition $(r_2 - r_1)/2 \leq r_1 \Lambda_2$, where $\Lambda_2 \approx 2,4929$. Therefore, if $M(A) \leq 0.28479$, then $c_2(A) = 1/4$.

Of course, the problem to find $c_2(A)$ is widely known. In addition, this paper is stimulated by the following question of A. I. Aptekarev: is it true that given $\lambda \in (0, 1/4)$ there exists a domain $\Omega_\lambda$ such that $c_2(\Omega_\lambda) = \lambda$? (raised in the seminar on Complex Analysis (Gonchar seminar), 14.11.2016). The main results of this paper are presented by Theorems 1 and 2, below. For any annulus $A$ with modulus $M(A) = M$ we prove that

$$c_2(A) = \begin{cases} 1/4, & \text{if } M \in (0, M^*]; \\ \gamma(2-\gamma)/4, & \text{if } M \in (M^*, \infty), \end{cases}$$

where $M^* \approx 0.57298$, $\gamma = \gamma(M) \in (1, 2)$, $\gamma((M^*, \infty)) = (1, 2)$. In particular, we get a positive answer to the Aptekarev question.

The paper is organized as follows. Using Gauss' hypergeometric functions in Section 2 we give equations to find the critical modulus $M^*$ and the values of the continuous function $\gamma : (M^*, \infty) \to (1, 2)$ and we formulate Theorems 1 and 2. Section 3 contains proofs of Theorems 1 and 2 via 5 lemmas. In Section 4 we examine the behavior of $c_2(A)$ as $M(A) \to \infty$ and describe an application of results to a Rellich type inequality.

## 2. Main results

We need the Euler gamma function $\Gamma$, the Gauss hypergeometric equation $\zeta(1-\zeta)\, u'' + (\gamma - (\alpha + \beta + 1)\, \zeta)\, u' - \alpha\beta\, u = 0$ and the hypergeometric series

$$F(\alpha, \beta; \gamma; \zeta) = 1 + \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)}{n!\,\Gamma(n+\gamma)}\, \zeta^n, \quad |\zeta| < 1,$$

with the following parameters: