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Asymptotic behaviour of the Sudler product of sines for quadratic irrationals

Sigrid Grepstad *, Mario Neumüller

Department of Financial Mathematics and Applied Number Theory, Johannes Kepler University Linz, Altenbergerstraße 69, 4040 Linz, Austria

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ABSTRACT

We study the asymptotic behaviour of the sequence of sine products $P_n(\alpha) = \prod_{r=1}^{n} |2\sin \pi r\alpha|$ for real quadratic irrationals α. In particular, we study the subsequence $Q_n(\alpha) = \prod_{r=1}^{q_n} |2\sin \pi r\alpha|$, where q_n is the nth best approximation denominator of α, and show that this subsequence converges to a periodic sequence whose period equals that of the continued fraction expansion of α. This verifies a conjecture recently posed by Verschueren and Mestel (2016) [15].

1. Introduction

In this paper, we study the sequence of sine products

$$P_n(\alpha) = \prod_{r=1}^{n} |2\sin \pi r\alpha|$$

for irrational $\alpha > 0$. Early studies of this product were conducted by Erdős and Szekeres [5] and Sudler [14] in the 1960s, and in following decades the sequence has proved important to both pure and applied mathematics (see e.g. [1,2,6], or [3,4,10,13] for a connection to q-series). It appears that research has been carried out simultaneously, and partly independently, in different mathematical disciplines, resulting in a number of different terminologies and representations of $P_n(\alpha)$. For a brief summary of key results we recommend the introduction of [15].

* Corresponding author. Present address: Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway.
E-mail addresses: sigrid.grepstad@ntnu.no (S. Grepstad), mario.neumueller@jku.at (M. Neumüller).

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The focus of this paper will be the subsequence

$$Q_n(\alpha) := \prod_{r=1}^{q_n} |2\sin \pi r\alpha|, \tag{1.1}$$

where $(q_n)_{n\geq 0}$ are the best approximation denominators of α. In a recent paper, Mestel and Verschueren study $Q_n(\alpha)$ in the special case where $\alpha = \omega := (\sqrt{5}-1)/2$ is the fractional part of the golden mean [15]. For this case, it was suggested by Knill and Tangerman in [7] that the limit value $\lim_{n\to\infty} Q_n(\omega)$ might exist, and this is confirmed by Mestel and Verschueren.

Theorem 1.1 *([15, Theorem 2.2]). If ω denotes the golden mean and $(F_n)_{n\geq 1} = (1, 1, 2, 3, 5, \ldots)$ the Fibonacci sequence, then there exists a constant $c > 0$ such that*

$$\lim_{n\to\infty} Q_n(\omega) = \lim_{n\to\infty} \prod_{r=1}^{F_n} |2\sin \pi r\omega| = c.$$

Mestel and Verschueren conjecture in [15] that Theorem 1.1 can be extended to all quadratic irrationals. More precisely, they suggest that if the continued fraction expansion of α has period ℓ, then the subsequence $Q_n(\alpha)$ will converge to a periodic sequence whose period length divides ℓ. Our main goal is to verify this claim.

Theorem 1.2. *Suppose α has a purely periodic continued fraction expansion $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$ with $a_1, \ldots, a_\ell \in \mathbb{N}$ and period ℓ. Let $(q_n)_{n\geq 1}$ be the sequence of best approximation denominators of α. Then there exist positive constants $C_0, C_1, \ldots, C_{\ell-1}$ such that*

$$\lim_{m\to\infty} Q_{\ell m+k}(\alpha) = \lim_{m\to\infty} \prod_{r=1}^{q_{\ell m+k}} |2\sin \pi r\alpha| = C_k$$

for each $k = 0, 1, 2, \ldots, \ell - 1$.

Corollary 1.3. *Suppose β has continued fraction expansion of the form $\beta = [a_0; a_1, \ldots, a_h, \overline{a_{h+1}, \ldots, a_{h+\ell}}]$ and let $\alpha = [0; \overline{a_{h+1}, \ldots, a_{h+\ell}}]$. We then have*

$$\lim_{m\to\infty} Q_{h+\ell m+k}(\beta) = \lim_{m\to\infty} Q_{\ell m+k}(\alpha).$$

The proof of Theorem 1.2 (and Corollary 1.3) largely follows that given by Mestel and Verschueren for the special case of the golden mean. Nevertheless, we include the proof in full detail for the sake of completeness. We emphasize that the challenge in generalizing Theorem 1.1 to all quadratic irrationals lies in finding appropriate analogs for $(q_n)_{n\geq 0}$ of certain special properties of the Fibonacci sequence $(F_n)_{n\geq 0} = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)$. Throughout their proof for the golden mean case, Mestel and Verschueren make heavy use of the identities

$$F_n\omega^n = \frac{1}{\sqrt{5}} + \mathcal{O}(\omega^{2n}) \quad (\text{for } n > 0)$$

and

$$\frac{F_{n-1}}{F_n} = \omega + \mathcal{O}(\omega^{2n}) \quad (\text{for } n > 0),$$