J. Math. Anal. Appl. 466 (2018) 986–1008

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

How far is the Borel map from being surjective in quasianalytic ultradifferentiable classes?

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A R T I C L E I N F O

Article history:
Received 13 March 2018
Available online 18 June 2018
Submitted by J. Bonet

Keywords:
Spaces of ultradifferentiable functions
Borel map
Quasianalyticity
Baire category
Prevalence
Lineability

A B S T R A C T

The Borel map j^∞ takes germs at 0 of smooth functions to the sequence of iterated partial derivatives at 0. In the literature, it is well known that the restriction of j^∞ to the germs of quasianalytic ultradifferentiable classes which are strictly containing the real analytic functions can never be onto the corresponding sequence space. In this paper, we are interested in studying how large the image of j^∞ is and we investigate the size and the structure of this image by using different approaches (Baire residuality, prevalence and lineability). We give an answer to this question in the very general setting of quasianalytic ultradifferentiable classes defined by weight matrices, which contains as particular cases the classes defined by a single weight sequence or by a weight function.

1. Introduction

In 1895, E. Borel proved that given any sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers, there exists a infinitely differentiable function such that $f^{(n)}(0) = a_n$ for every $n \in \mathbb{N}$ [8]. This work has been investigated and extended ever since by many authors. In particular, the question has been handled in the context of so-called ultradifferentiable classes which are subclasses of smooth functions defined by imposing growth conditions on the derivatives of the functions using weight sequences M, functions ω or matrices $\mathcal{M}$, see [10,11,17,26,7,6,4,20].

Historically, those classes have been first introduced by using weight sequences, motivated among others by the characterization of the regularity of solutions of the heat equation or of other partial differential equations, see e.g. [21]. In order to measure the decay of the Fourier transform of smooth functions with compact support, classes of ultradifferentiable functions have then been defined using weight functions,

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https://doi.org/10.1016/j.jmaa.2018.06.037
0022-247X/

e.g. see [3] and [18]. In [9], it turned out that such a behavior can also equivalently be expressed by having control on the growth of all the derivatives of the function itself in terms of this weight function and in [5] it has been shown that classes defined in terms of weight sequences and weight functions are in general mutually distinct. Finally, in [19] and [24], classes defined by weight matrices have been considered. It turned out that the weight sequence and weight function frameworks are particular cases of this setting, and this general method allows to treat both classical approaches jointly but also leads to more general classes.

We say that an ultradifferentiable class is *quasianalytic* if the restriction of the Borel map $f \mapsto (\partial^\alpha f(0))_{\alpha\in\mathbb{N}^r}$ to this class is injective; this notion plays an important role in many different contexts and applications (e.g. such classes do not contain partitions of unity). It came out of many studies that the restriction of the Borel map to the germs of quasianalytic ultradifferentiable classes which are strictly containing the real analytic functions can never be onto the corresponding sequence space. However, an interesting remaining question is "how far away the Borel map is from being surjective?" This is the question we tackle in this paper: We show that the image of the Borel map is "small" in the corresponding sequence space, using different approaches (as done e.g. in [13]). Let us present these different notions here.

First, let us recall the following classical definition which gives a notion of residuality from a topological point of view.

Definition 1.0.1. If X is a Baire space, then a subset $L \subset X$ is called *comeager* (or *residual*) if L contains a countable intersection of dense open sets of X. The complement of a residual set is a *meager* (or *first category*) set in X.

In order to get result about the "size" of sets from a measure-theoretical point of view, the notion of prevalence can be used. It has been introduced in [12,14] to give an extension of the concept of "almost everywhere" (for the Lebesgue measure) to metric infinite dimensional spaces (in these spaces, no measure is both σ-finite and translation invariant).

Definition 1.0.2. Let X denote a complete metric vector space. A Borel subset $B \subset X$ is called *Haar-null* if there exists a compactly supported probability measure μ such that

$$\forall x \in X, \quad \mu(x+B) = 0. \tag{1.1}$$

A subset S of X is called *Haar-null* if it is contained in a Haar-null Borel set. A *prevalent* set is the complement of a Haar-null set.

The following results of [12] and [14] enumerate important basic properties of prevalent sets:

- If S is Haar-null, then $x + S$ is Haar-null for any $x \in X$.
- If the dimension of X is finite, S is Haar-null if and only if S has Lebesgue measure 0.
- Prevalent sets are dense.
- Any countable intersection of prevalent sets is prevalent.

Remark 1.0.3. A useful way to get that a Borel set is Haar-null is to try the Lebesgue measure on the unit ball of a finite dimensional subspace V. In this context, condition (1.1) is equivalent to

$$\forall x \in X, \quad (x+B)\cap V \text{ is of Lebesgue measure zero.}$$

In this case, we say that V is a *probe* for the complement of B.

Finally, we will also consider the notion of *lineability*, introduced in [1]. This notion was motivated by the increasing interest toward the search for large algebraic structures of special objects (see [2] for a review).