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# On harmonic numbers and nonlinear Euler sums

Ce Xu [a,*], Yulin Cai [b]

[a] *School of Mathematical Sciences, Xiamen University, Xiamen 361005, PR China*
[b] *Institut de Mathematiques de Bordeaux, Universite de Bordeaux, Bordeaux 33405, France*

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A B S T R A C T

In this paper we are interested in Euler-type sums with products of harmonic numbers, Stirling numbers and Bell numbers. We discuss the analytic representations of Euler sums through values of polylogarithm function and Riemann zeta function. Moreover, we provide explicit evaluations for almost all Euler sums with weight $\leq 5$, which can be expressed in terms of zeta values and polylogarithms. Furthermore, we give explicit formula for several classes of Euler-related sums in terms of zeta values and harmonic numbers, and several examples are given. The given representations are new.



**Contents**



* Corresponding author.
*E-mail addresses:* xuce1242063253@163.com (C. Xu), Yulin.Cai@math.u-bordeaux.fr (Y. Cai).

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## 1. Introduction and preliminaries

### 1.1. Harmonic numbers and Euler sums

The generalized $n$th-harmonic numbers of order $p$ are given by ([23])

$$H_n^{(p)} := \sum_{j=1}^{n} \frac{1}{j^p}, \quad p \in \mathbb{N}, \tag{1.1}$$

where the $H_n \equiv H_n^{(1)}$ is the classical harmonic number ([1]) and the empty sum $H_0^{(p)}$ is conventionally understood to be zero.

In response to a letter from Goldbach in 1742, Euler considered sums of the form (see Berndt [4])

$$S_{p,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q},$$

where $p, q$ are positive integers with $q \geq 2$, and $w := p + q$ denotes the weight of sums $S_{p,q}$. These sums are called the linear Euler sums today. Euler discovered that in the cases $p = 1,\ p = q$ and $p + q$ is less than 7 or when $p + q$ is odd and less than 13, the linear sums have evaluations in terms of zeta values, i.e., the values of the Riemann zeta function ([2])

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \Re(s) > 1$$

at the positive integer arguments. Moreover, he conjectured that the double linear sums would be reducible to zeta values when $p + q$ is odd, and even gave what he hoped was the general formula. In 1995, Borweins and Girgensohn [10] proved the conjecture and formula, and in 1994, Bailey, Borwein and Girgensohn [3] conjectured that the linear sums $S_{p,q}$, when $p + q > 7$ and $p + q$ is even, are not reducible. Hence, the linear sums $S_{p,q}$ can be evaluated in terms of zeta values in the following cases: $p = 1, p = q, p + q$ odd and $p + q = 4, 6$ with $q \geq 2$.

Similarly, the nonlinear Euler sums are the infinite sums whose general term is a product of harmonic numbers of index $n$ and a power of $n^{-1}$. Let $\pi := (\pi_1, \ldots, \pi_k)$ be a partition of integer $p$ and $p = \pi_1 + \cdots + \pi_k$ with $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_k$. The classical nonlinear Euler sum of index $\pi, q$ is defined as follows (see [23])

$$S_{\pi_1\pi_2\cdots\pi_k,q} := \sum_{n=1}^{\infty} \frac{H_n^{(\pi_1)} H_n^{(\pi_2)} \cdots H_n^{(\pi_k)}}{n^q}, \quad q \geq 2 \tag{1.2}$$

where the quantity $w := \pi_1 + \cdots + \pi_k + q$ is called the weight and the quantity $k$ is called the degree. As usual, repeated summands in partitions are indicated by powers, so that for instance

$$S_{1^2 2^3 4,q} = S_{112224,q} = \sum_{n=1}^{\infty} \frac{H_n^2 [H_n^{(2)}]^3 H_n^{(4)}}{n^q}.$$

The nonlinear Euler sums, i.e., $S_{\pi,q}$ with $\pi$ having two or more parts, are more complicated. Such sums were already considered in [3,23,43,45,47,48,51,52]. In [23], Flajolet and Salvy gave an algorithm for reducing $S_{\pi_1\pi_2,q}$ to linear Euler sums when $\pi_1 + \pi_2 + q$ is even and $\pi_1, \pi_2, q > 1$ (see Theorem 4.2 in the reference [23]). In [51,52], the first author jointly with Li, Yan and Shi proved that all quadratic Euler sums of the form