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Rearrangements and Leibniz-type rules of mean oscillations ☆

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We shall prove a rearrangement inequality in probability measure spaces in order to obtain sharp Leibniz-type rules of mean oscillations in L^p-spaces and rearrangement invariant Banach function spaces.

1. Introduction

Rearrangements and rearrangement inequalities are powerful tools in functional analysis. One can find applications, for instance, in symmetry problems in the calculus of variations, interpolation theory or matrix analysis as well, see e.g. [13], [1] and [3].

Our aim is to provide a rearrangement inequality, which seems to have been unnoticed so far, in the style of the classical Hardy–Littlewood inequality. In addition, we shall use the result to offer a simple new proof of sharp Leibniz-type rules of mean oscillations in L^p-spaces and an extension to rearrangement invariant function spaces.

In general, Leibniz-type rules quantify the seminorms of products in function or operator algebras in terms of the (semi)norms of their factors. To be more precise, one may consider inequalities

$$\|fg\|_Z \lesssim \|g\|_{X_1}\|f\|_{Y_1} + \|g\|_{X_2}\|f\|_{Y_2},$$

for all f, g in the space Z, with appropriate (semi)norms X_1, X_2, Y_1, Y_2. However, to determine the sharp constant of the right-hand side, or to prove whether or not it is finite, depending on the spaces X_1, X_2, Y_1, Y_2, can be a challenging problem.

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Sharp Leibniz inequalities and Leibniz seminorms have appeared in M. Rieffel's fundamental work on quantum metric spaces, see e.g. [15], [17], [18], [16]. Briefly, we say that a seminorm L is Leibniz on a unital algebra $(\mathcal{A}, \|\cdot\|)$ if it vanishes on the unit element of $\mathcal{A}$ and $L(ab) \le \|b\|L(a) + \|a\|L(b)$ is satisfied for all $a, b \in \mathcal{A}$. For instance, if (X, d) denotes a compact metric space then

$$\mathrm{Lip}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\},$$

the Lipschitz constant of a continuous function $f : X \to \mathbb{C}$, defines such a seminorm on $C(X)$, the algebra of continuous functions over X endowed with the usual sup norm ($\mathrm{Lip}(f) = \infty$ may happen). Interestingly, one can simply recover the underlying metric d on X through Lip and, roughly speaking, the metric data of non-commutative compact C^*-metric spaces can be encoded by Leibniz seminorms as well. Such seminorms are naturally arising from first-order differential calculi, inner derivations while others are coming from ergodic actions of compact groups, see e.g. [16, Section 2]. Another simple (but non-trivial) example is the standard deviation defined in ordinary and non-commutative probability spaces [18]. From a broader perspective, Dirichlet forms on real L^2-spaces define Leibniz seminorms as well. Indeed, if $\mathcal{E}$ is a Dirichlet form with its domain $D(\mathcal{E})$ then

$$\sqrt{\mathcal{E}(fg, fg)} \le \|f\|_\infty \sqrt{\mathcal{E}(g, g)} + \|g\|_\infty \sqrt{\mathcal{E}(f, f)}$$

holds for all $f, g \in L^\infty \cap D(\mathcal{E})$, see e.g. [8, Theorem 1.4.2], [4, Corollary 3.3.2]. In fact, any Dirichlet form can be represented as a quadratic form associated to a closable derivation, which may serve as a direct link to Leibniz-type inequalities, see [6] and the references therein.

On the other hand, we have to admit that the Kato–Ponce inequalities are likely to be the most well-known Leibniz-type inequalities. We just recall the result in the following form: let $(-\Delta)^\alpha$ be the fractional Laplacian defined as the Fourier multiplier

$$\widehat{(-\Delta)^\alpha} f(\xi) = |\xi|^{2\alpha} \hat{f}(\xi), \quad \xi \in \mathbb{R}^n,$$

for any f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. If $1 < r, p_1, q_1, p_2, q_2 < \infty$ such that $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ and $0 < \alpha \le 1$, one has, for all $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\|(-\Delta)^\alpha (fg)\|_r \le C(\|g\|_{p_1} \|(-\Delta)^\alpha f\|_{q_1} + \|f\|_{p_2} \|(-\Delta)^\alpha g\|_{q_2}),$$

where $C = C_{n,\alpha,p_1,q_1,p_2,q_2,r} > 0$ is a constant depending only on $(n, \alpha, p_1, q_1, p_2, q_2, r)$. Nowadays, the Kato–Ponce inequalities have been extensively studied and have a large literature. We refer the reader to [9] and [14], for instance.

In this paper, we shall provide Leibniz-type rules of mean oscillations via rearrangement inequalities. We would like to convince the reader that rearrangements of functions naturally appear when we discuss these inequalities. The present paper is a continuation of the recent ones [2], [11], [12] and provides a completely different view on our earlier results with an extension to rearrangement invariant function spaces.

Given a probability space $(\Omega, \mathcal{F}, \mu)$, suppose $f, g \in L^\infty(\Omega, \mu)$ and $h \in L^1(\Omega, \mu)$ are real-valued μ-measurable functions. First, we shall prove a rearrangement inequality in Theorem 3.3 below. There the mean-zero condition

$$\int_\Omega g \, d\mu = 0$$

turns out to be crucial to obtain the inequality