J. Math. Anal. Appl. 466 (2018) 1043–1059

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A Coburn type theorem on the Hardy space of the bidisk [☆]

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ARTICLE INFO

Article history:
Received 7 January 2018
Available online 18 June 2018
Submitted by Z. Cuckovic

Keywords:
Toeplitz operators
Hardy space
Bidisk

ABSTRACT

A famous theorem of Coburn says that a nonzero Toeplitz operator on the Hardy space of the unit disk is injective or its adjoint operator is injective. In this paper we study the corresponding problem on the Hardy space of the bidisk. We first show the Coburn type theorem fails generally on the bidisk. But, we show that certain pluriharmonic symbols or product symbols of one variable functions induce Toeplitz operators satisfying the Coburn type theorem.

1. Introduction

Let $\mathbb{T}$ be the boundary of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. The bidisk $\mathbb{D}^2$ and torus $\mathbb{T}^2$ are the cartesian products of 2 copies of $\mathbb{D}$ and $\mathbb{T}$ respectively. We let $L^p(\mathbb{T}^2) = L^p(\mathbb{T}^2, \sigma)$ denote the usual Lebesgue space on $\mathbb{T}^2$ where $\sigma = \sigma_2$ is the normalized Haar measure on $\mathbb{T}^2$. The Hardy space $H^2(\mathbb{D}^2)$ is the closure of the holomorphic polynomials in $L^2(\mathbb{T}^2)$. As is well known, we can identify a function in $H^2(\mathbb{D}^2)$ with its holomorphic extension to $\mathbb{D}^2$ via the Poisson extension. Thus, we will use the same notation for a function $f \in H^2(\mathbb{D}^2)$ and its holomorphic extension f on $\mathbb{D}^2$. Let P denote the orthogonal projection from $L^2(\mathbb{T}^2)$ onto $H^2(\mathbb{D}^2)$. For a function $u \in L^\infty(\mathbb{T}^2)$, the Toeplitz operator T_u with symbol u is defined by

$$T_u f = P(uf)$$

for functions $f \in H^2(\mathbb{D}^2)$. Then clearly T_u is a bounded linear operator on $H^2(\mathbb{D}^2)$.

On the Hardy space of the unit disk, a celebrated theorem of Coburn asserts that for a nonzero Toeplitz operator, we have either it is injective or its adjoint operator is injective. This theorem is implicit in the

☆ The first author was supported by NSFC (11771401) and ZJNSFC (LY14A010013) and the second author is supported by JSPS KAKENHI Grant Number 15K04895. Also the third author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A3B03933949).
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https://doi.org/10.1016/j.jmaa.2018.06.034
0022-247X/

proof of Theorem 4.1 of [1]. See also Proposition 7.2.4 of [2]. Later, Vukotić [6] reproved the theorem by making the statement above more explicit by showing that the range of a nonzero Toeplitz operator which is not injective contains the set of all analytic polynomials.

In this paper we naturally consider the corresponding problem for Toeplitz operators acting on the Hardy space of the bidisk. First of all, we should mention that the Coburn type theorem fails generally on the bidisk. For an example, one can see

$$T_{z^2\overline{w}^2}(z) = 0 = T^*_{z^2\overline{w}^2}(z)$$

on $H^2(\mathbb{D}^2)$ where S^* denotes the adjoint operator of a bounded operator S. Thus the Toeplitz operator $T_{z^2\overline{w}^2}$ doesn't satisfy the Coburn type theorem; see Section 2 for details. On the other hand, we can easily see that a Toeplitz operator with a nonzero (anti-)holomorphic symbol satisfies the Coburn type theorem on $H^2(\mathbb{D}^2)$. Also, one can check that a Toeplitz operator induced by a symbol depending only one variable satisfies the Coburn type theorem on $H^2(\mathbb{D}^2)$; see Corollary 16. In view of this observation, we naturally pose the following problem: *For which symbol, does the corresponding Toeplitz operator satisfy the Coburn type theorem on $H^2(\mathbb{D}^2)$?*

Motivated by examples mentioned above, we consider in this paper three classes of symbols as outlined below:

(1) symbols of the form $u = \varphi + \overline{\psi}$ where φ, ψ are bounded holomorphic on $\mathbb{D}^2$ and
    (a) φ is bounded by 1 and ψ is inner,
    or (b) $\varphi = \varphi(z)$ is general and $\psi = \psi(w)$ is inner,
    or (c) ψ is not assumed to be inner.
(2) symbols of the form $u = f(z)g(w)$ where f, g are bounded on $\mathbb{T}$.
(3) symbols of the form $u = \sum_{j=0}^{\infty} \overline{h_j}(z)w^j$ where h_j is holomorphic on $\mathbb{D}$.

We then provide several sufficient conditions on the symbols u for which T_u satisfies the Coburn type theorem on $H^2(\mathbb{D}^2)$. More explicitly, we first in Section 3 consider pluriharmonic symbols of the form $\varphi + \overline{\psi}$ where $\varphi \in H^\infty(\mathbb{D}^2)$ and ψ is a non-constant inner function. Here the space $H^\infty(\mathbb{D}^2)$ denotes the space of all bounded holomorphic functions on $\mathbb{D}^2$ and we write $\|\varphi\|_\infty$ for the essential supremum norm for a function $\varphi \in L^\infty(\mathbb{D}^2)$. Also we say that a function in $H^\infty(\mathbb{D}^2)$ is called inner if the modulus of its radial limit is equal to 1 a.e. on $\mathbb{T}^2$; see [5] for details. For such a pluriharmonic symbol $u = \varphi + \overline{\psi}$, if $\|\varphi\|_\infty < 1$, then we first show that T_u^* is injective but T_u is not injective. Moreover we describe the kernel of T_u; see Theorem 2.

Specially, for symbols of the form $u = \varphi(z) + \overline{\psi}(w)$ where φ and ψ depend on a different single variable, we show that the boundedness condition $\|\varphi\|_\infty < 1$ can be removed. More explicitly, we characterize the injectivity of T_u and then, as an application, we show that the corresponding Toeplitz operator satisfies the Coburn type theorem on $H^2(\mathbb{D}^2)$; see Theorem 3 and Corollary 4.

We also consider general ψ other than inner functions. For such symbols $u = \varphi(z)+\overline{\psi}(w)$, if $\|\varphi\|_\infty \neq \|\psi\|_\infty$ or $\|\varphi\|_\infty = \|\psi\|_\infty = 1$ together with certain boundary conditions, then we show that T_u satisfies the Coburn type theorem; see Theorems 5 and 9. But we don't know whether a Toeplitz operator with general pluriharmonic symbol satisfies the Coburn type theorem on $H^2(\mathbb{D}^2)$.

Next, in Section 4 we consider symbols which are products of two one variable functions depending on a different single variable. Namely, we consider symbols of the form $u = f(z)g(w)$ where $f, g \in L^\infty(T)$ are nonzero functions. For such a symbol, we first describe the kernel of T_u and then, as immediate consequences, obtain several kinds of symbols of Toeplitz operators satisfying the Coburn type theorem; see Theorem 13 and its corollaries.