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Closed-form expressions for derivatives of Bessel functions with respect to the order

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ABSTRACT

We have used recent integral representations of the derivatives of the Bessel functions with respect to the order to obtain closed-form expressions in terms of generalized hypergeometric functions and Meijer-G functions. Also, we have carried out similar calculations for the derivatives of the modified Bessel functions with respect to the order, obtaining closed-form expressions as well. For this purpose, we have obtained integral representations of the derivatives of the modified Bessel functions with respect to the order. As by-products, we have calculated two non-tabulated integrals.

1. Introduction

The Bessel functions have had many applications since F.W. Bessel (1784–1846) found this kind of functions in his studies of planetary motion. In Physics, these functions arise naturally in boundary value problems of potential theory for cylindrical domains [8, Chap. 6]. In Mathematics, the Bessel functions are encountered in the theory of differential equations with turning points, and well as with poles [11, Sect. 10.72]. Thereby, the theory of Bessel functions has been studied extensively in many classical textbooks [14,1].

Usually, the definition of the Bessel function of the first kind $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$ are given in series form as follows:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!\,\Gamma(\nu+k+1)}, \tag{1}$$

and

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$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!\,\Gamma(\nu+k+1)}. \tag{2}$$

The Bessel function of the second kind $Y_\nu(z)$ is defined in terms of the Bessel function of the first kind as

$$Y_\nu(z) = \frac{J_\nu(z)\cos\pi\nu - J_{-\nu}(z)}{\sin\pi\nu}, \qquad \nu \notin \mathbb{Z}, \tag{3}$$

and similarly, for the Macdonald function $K_\nu(z)$, we have

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\pi\nu}, \qquad \nu \notin \mathbb{Z}. \tag{4}$$

Despite the fact, that the literature about the Bessel functions is very large as mentioned before, the literature regarding the derivatives of J_ν, Y_ν, I_ν and K_ν with respect to the order ν is relatively scarce. For instance, for $\nu = \pm 1/2$ we find expressions for the order derivatives in terms of the exponential integral $\mathrm{Ei}(z)$ and the sine and cosine integrals, $\mathrm{Ci}(z)$ and $\mathrm{Si}(z)$ [10,3]. By using the recurrence relations of Bessel [11, Eqn. 10.6.1] and modified Bessel functions [8, Eqn. 5.7.9], we can derive expressions for half-integral order $\nu = n \pm 1/2$. Also, for integral order $\nu = n$ we find some series representations in [3]. For arbitrary order, we have the following series representations [11, Eqns. 10.15.1 & 10.38.1]

$$\frac{\partial J_\nu(z)}{\partial\nu} = J_\nu(z)\log\left(\frac{z}{2}\right) - \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\psi(\nu+k+1)(-1)^k (z/2)^{2k}}{k!\,\Gamma(\nu+k+1)}, \tag{5}$$

and

$$\frac{\partial I_\nu(z)}{\partial\nu} = I_\nu(z)\log\left(\frac{z}{2}\right) - \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\psi(\nu+k+1)(z/2)^{2k}}{k!\,\Gamma(\nu+k+1)}, \tag{6}$$

which are obtained directly from (1) and (2). Also, from (3) and (4), we can calculate the order derivative of Y_ν and K_ν as [11, Eqns. 10.15.2 & 10.38.2],

$$\frac{\partial Y_\nu(z)}{\partial\nu} = \cot\pi\nu\left[\frac{\partial J_\nu(z)}{\partial\nu} - \pi\, Y_\nu(z)\right] - \csc\pi\nu\,\frac{\partial J_{-\nu}(z)}{\partial\nu} - \pi\, J_\nu(z), \tag{7}$$

and

$$\frac{\partial K_\nu(z)}{\partial\nu} = \frac{\pi}{2}\csc\pi\nu\left[\frac{\partial I_{-\nu}(z)}{\partial\nu} - \frac{\partial I_\nu(z)}{\partial\nu}\right] - \pi\cot\pi\nu\, K_\nu(z). \tag{8}$$

Although we can accelerate the convergence of the alternating series given in (5) by using Cohen–Villegas–Zagier algorithm [5], this series does not converge properly for large z, and it is not useful from a numeric point of view. Also, the series given in (6) is not useful for large z as well.

Nonetheless, in the literature we find integral representations of $J_\nu(z)$ and $I_\nu(z)$ in [2], which read as

$$\frac{\partial J_\nu(z)}{\partial\nu} = \pi\nu \int_0^{\pi/2} \tan\theta\; Y_0\left(z\sin^2\theta\right) J_\nu\left(z\cos^2\theta\right) d\theta, \qquad \mathrm{Re}\,\nu > 0, \tag{9}$$

and

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