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Holographic coordinates

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ABSTRACT

The Laplace equation in the two-dimensional Euclidean plane is considered in the context of the inverse stereographic projection. The Lie algebra of the conformal group as the symmetry group of the Laplace equation can be represented solely in terms of the solutions and derivatives of the solutions of the Laplace equation. It is then possible to put contents from differential geometry and quantum systems, like the Hopf bundle, relativistic spin, bicomplex numbers, and the Fock space into a common context. The basis elements of the complex numbers, considered as a Clifford paravector algebra, are reinterpreted as differential tangent vectors referring to dilations and rotations. In relation to this a homogeneous space is defined with the Lie algebra of the conformal group, where dilations and rotations are the coset representatives. Potential applications in physics are discussed.

1. Introduction

The dimensional reduction of Minkowski space has been discussed more than two decades ago by 't Hooft [93]. These investigations lead to what is known today as the holographic principle [91]. It has been suggested that the holographic principle is a foundation of a quantum gravity theory in analogy to the equivalence principle as a foundation of general relativity [18]. The holographic principle influenced string theory and lead to the AdS/CFT correspondence [34,59,109]. The holographic principle is applied in different physical areas like for example loop quantum gravity, cosmology, QCD, condensed matter physics, instantons, and neutron stars [6,19,37,38,40,41,43,46,62,76,81–83,92].

In accordance with the holographic principle one can see projective mappings as geometric operations, which transform within representation spaces of different dimension. Here Möbius geometries and projective Lorentz spaces play a central role. For the mathematical background with respect to Möbius geometries it is referred to Hertrich-Jeromin [39], Sharpe [84], Kisil [49,50], or Jensen et al. [45]. Furthermore, a hierarchy of projective geometric spaces and Möbius geometries can be introduced based on Vahlen matrices [1,2,63,98]. Maks [58] investigated explicitly the sequence of Möbius geometries represented in terms of Clifford algebras,

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which is considered also in [97] in a bicomplex matrix representation. In relation to particle physics one may assume that projections of a more general geometry can arise as part of the measurement process via electromagnetic forces. In accordance with the holographic principle Möbius geometries and Möbius transformations can provide here a new perspective on the experimental data.

Atiyah, Manton, and Schroers [8–11] describe electrons, protons, neutrinos, and neutrons within a model that has been inspired by Skyrme's baryon theory [86]. The Skyrme model became popular when Witten came up with the idea that baryons arise as solitons of the classical meson fields [107]. The Skyrme theory turned out to be remarkably successful in describing baryons in the previous decades [75]. Thus matter is supposed to arise from a pure geometric foundation [24,60]. Solitons are usually considered in the context of gauge transformations, which relate in a mathematical context to fiber bundles and Cartan geometries [84].

With regards to the topics discussed above research in two-dimensional conformal field theories [15,16,32,48,65] has turned out to be important, also because conceptual insights can be obtained more easily in a lower dimensional geometry. Motivation to proceed in this direction is also provided by [97]. In this article the two-dimensional Euclidean plane and the conformal symmetries, which refer to the Laplace equation in R^2, are fundamental in a sequence of higher dimensional geometries and Clifford algebras. Therefore, one can ask the question how the Laplace equation can be embedded into higher dimensional geometries. Here concepts like the stereographic projection should play an important role. In this sense one can consider the Laplace equation in the base space S^2 instead of R^2 by means of what is denoted here as holographic coordinates. This conceptual change is adopted from Lie sphere geometry [21,45], where the spherical geometry is leading to considerable simplifications compared to the Euclidean geometry.

The main result of this article is that with the help of the holographic coordinates the Lie algebra of the conformal group in two dimensions can be represented solely in terms of the solutions and derivatives of the solutions of the corresponding Laplace equation. It turns out that different concepts in mathematics and physics, like projective spaces, spin, the Hopf bundle, Minkowski space, or bicomplex numbers can be seen in relation to each other. The connections are provided by the solutions and symmetries of the Laplace equation.

2. Conformal transformations

Conformal transformations have been investigated in [97] based on spin representations over the ring of bicomplex numbers. The non-zero commutation relations for these spin matrices are

$$\begin{aligned} [s_{\mu\nu}, p_\sigma] &= g_{\nu\sigma} p_\mu - g_{\mu\sigma} p_\nu \,, \\ [s_{\mu\nu}, q_\sigma] &= g_{\nu\sigma} q_\mu - g_{\mu\sigma} q_\nu \,, \\ [b, p_\mu] &= -p_\mu \,, \\ [b, q_\mu] &= q_\mu \,, \\ [q_\mu, p_\nu] &= 2(g_{\mu\nu} b + s_{\mu\nu}) \,. \end{aligned} \tag{1}$$

Here $s_{\mu\nu}$ denotes the angular momentum operator, p_μ the momentum, and q_μ the special conformal transformations. Compared to [97] the symbol b instead of d is used to label the scale transformation in order to avoid future confusion with the exterior derivative. The commutation relation between two angular momentum operators is not listed here. It will be discussed separately later in the text.

The difference of the matrix commutation relations compared to the relations used by Kastrup [47] is that no complex units appear on the right-hand side of the equations. If one tries to represent the spin matrices in terms of differential operators, one cannot apply the conventions of quantum physics and add complex units to the differential operators. However, one can start in the context of classical field theory and use the conventions of Freedman and Van Proeyen [29], which result in the following representation

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