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Extremes of vector-valued Gaussian processes with Trend

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ABSTRACT

Let $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t)), t \in \mathcal{T} \subset \mathbb{R}$ be a centered vector-valued Gaussian process with independent components and continuous trajectories, and $\boldsymbol{h}(t) = (h_1(t), \ldots, h_n(t)), t \in \mathcal{T}$ be a vector-valued continuous function. We investigate the asymptotics of

$$\mathbb{P}\left\{\sup_{t\in\mathcal{T}} \min_{1\le i\le n} (X_i(t) + h_i(t)) > u\right\}$$

as $u \to \infty$. As an illustration to the derived results we analyze two important classes of $\boldsymbol{X}(t)$: with locally-stationary structure and with varying variances of the coordinates, and calculate exact asymptotics of simultaneous ruin probability and ruin time in a Gaussian risk model.

1. Introduction and preliminaries

Motivated by various applied-oriented problems, the asymptotics of

$$\mathbb{P}\left\{\sup_{t\in\mathcal{T}}(X(t) + h(t)) > u\right\}, \tag{1}$$

as $u \to \infty$, for both $\mathcal{T} = [0, T]$ and $\mathcal{T} = [0, \infty)$, where $X(t)$ is a centered Gaussian process with continuous trajectories and $h(t)$ is a continuous function, attracted substantial interest in the literature; see e.g. [29,9,30,22,27,19,12,13] and references therein for connections of (1) with problems considered, e.g., in risk theory or fluid queueing models. For example, in the setting of risk theory one usually supposes that $h(t) = -ct$, with $c > 0$ and X has stationary increments. Then, using that $\mathbb{P}\{\sup_{t\in\mathcal{T}}(X(t) + h(t)) > u\} = \mathbb{P}\{\inf_{t\in\mathcal{T}}(u - X(t) + ct) < 0\}$, (1) represents *ruin probability*, with $X(t)$ modelling the accumulated claims amount in time interval $[0, t]$, c being the constant premium rate and u, the initial capital. The most

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celebrated model in this context is the Brownian risk model introduced in the seminal work by Iglehart [33], where X is a standard Brownian motion. Extensions to more general class of Gaussian processes with stationary increments, including fractional Brownian motions, was analyzed in, e.g., [34,29,30,32,31]. Recent interest in the analysis of risk models has turned to the investigation of multidimensional ruin problems, including investigation of *simultaneous ruin* probability of some number, say n, of independent risk processes

$$\mathbb{P}\left\{\exists_{t\in\mathcal{T}}\forall_{i=1,\dots,n}(u_i - X_i(t)+c_i t)<0\right\},$$

see, e.g., [2] and [3]. Motivated by this sort of problems, in this paper we investigate multidimensional counterpart of (1), i.e., we are interested in the exact asymptotics of

$$\mathbb{P}\left\{\exists_{t\in[0,T]}\boldsymbol{X}(t)+\boldsymbol{h}(t)>u\mathbf{1}\right\} = \mathbb{P}\left\{\sup_{t\in[0,T]}\min_{1\le i\le n}(X_i(t)+h_i(t))>u\right\}, \tag{2}$$

as $u\to\infty$, $T\in(0,\infty)$, where $\boldsymbol{X}(t)=(X_1(t),\dots,X_n(t)), t\in\mathcal{T}\subset\mathbb{R}$ is an n-dimensional centered Gaussian process with mutually independent coordinates and continuous trajectories and $\boldsymbol{h}(t)=(h_1(t),\dots,h_n(t)), t\in[0,T]$ is a vector-valued continuous function.

The main results of this contribution extend recent findings of [16], where the exact asymptotics of (2) for $h_i\equiv 0, 1\le i\le n$ was analyzed; see also [21] where $\boldsymbol{X}(t)$ is a multidimensional Brownian motion, $h_i(t)=c_i t$ and $T=\infty$, and [14,40] for LDP-type results. It appears that the presence of the drift function substantially increases difficulty of the problem when comparing it with the analysis given for the driftless case in [16]. More specifically, as advocated in Section 2, it requires to deal with

$$\mathbb{P}\left\{\sup_{t\in[0,T]}\min_{1\le i\le n}X_{u,i}(t)>u\right\}, \tag{3}$$

where $(X_{u,i}(t), t\in[0,T])_u$, $i=1,...,n$ are families (with respect to u) of centered threshold-dependent Gaussian processes; see Theorem 2.1 which is the main result of this contribution. The proof of Theorem 2.1 is based on a refinement of the *double sum* method, a technique which was originally developed by Pickands [35] for centered stationary Gaussian processes and generalized by Piterbarg and Prisjažnjuk [39] to nonstationary setup; see also the seminal monograph [37] and [38]. The idea of the proof is based on an appropriate division of the parameter set $[0,T]$ onto tiny intervals so that the (conditioned) process converges to some limit process and then the analysis of the probability under the study over those intervals; see Lemma 4.1. The second main step in the proof is to show that (3) asymptotically behaves like the sum of the probabilities over the chosen subintervals; combination of Bonferroni inequality with Lemma 4.2 is crucial here.

In Section 3 we apply general results derived in Section 2 to two important families of Gaussian processes, i.e. i) to locally-stationary processes in the sense of Berman and ii) to processes with varying variance $\mathrm{Var}(X_i(t))$, $t\in[0,T]$. Then, as an example to the derived theory, we analyze the probability of simultaneous ruin in Gaussian risk model. Complementary, we investigate the limit distribution of the *simultaneous ruin time*

$$\tau_u := \inf\{t\ge 0 : (\boldsymbol{X}(t)+\boldsymbol{h}(t))>u\mathbf{1}\},$$

conditioned that $\tau_u\le T$, as $u\to\infty$.

Organization of the rest of the paper: Section 2 is devoted to the main result of this contribution, concerning the extremes of the threshold-dependent centered Gaussian vector processes. In Section 3 we specify our result to locally-stationary vector-valued Gaussian processes with trend and non-stationary