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Spectral properties of tensor products of channels

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ABSTRACT

We investigate spectral properties of the tensor products of two completely positive and trace preserving linear maps (also known as quantum channels) acting on matrix algebras. This leads to an important question of when an arbitrary subalgebra can split into the tensor product of two subalgebras. We show that for two unital quantum channels the multiplicative domain of their tensor product splits into the tensor product of the individual multiplicative domains. Consequently, we fully describe the fixed points and peripheral eigen operators of the tensor product of channels. Through a structure theorem of maximal unital proper *-subalgebras (MUPSA) of a matrix algebra we provide a non-trivial upper bound of the recently-introduced multiplicative index of a unital channel. This bound gives a criteria on when a channel cannot be factored into a product of two different channels. We construct examples of channels which cannot be realized as a tensor product of two channels in any way. With these techniques and results, we found some applications in quantum information theory.

1. Introduction

If we have a linear map acting on a matrix algebra that can be expressed as a tensor product of matrix algebras, and the map itself can be expressed as a tensor product of two other linear maps, there may be few similarities between the constituent maps and the larger linear map they produce. If we restrict ourselves to special classes of linear maps and special domains of matrix algebras, then the tensor product adds no extra complexity. Our goal in this paper is to use the multiplicative domain to characterize some of these properties for trace-preserving, completely positive maps on matrix algebras. These maps are also known as quantum channels, which we refer to as channels.

The multiplicative domain of a linear map $\mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d$ is the set of all matrices $x \in \mathcal{M}_d$ such that, for all $y \in \mathcal{M}_d$, $\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y)$ and $\mathcal{E}(yx) = \mathcal{E}(y)\mathcal{E}(x)$. When the linear map is completely positive,

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this specific set has received much attention in operator theory and operator algebras ([6], [26, chapter 4], [30, section 2.1]). In this context, it characterizes certain distinguishability measures. A completely positive linear map acts like a homomorphism on the multiplicative domain, and hence studying this domain can reveal structure and properties of the linear map.

In quantum information theory ([19], [7], [23]), the multiplicative domain contains the unitarily correctable codes and noiseless subsystems. Studying the multiplicative domain of tensor products sheds light on error correction in bipartite systems.

It turns out that we can capture most of the spectral properties of the tensor product of channels simply by investigating the multiplicative behavior. Note that the spectral properties of a channel acting on one copy of a quantum system have been well explored ([34], [3], [8], [35]) for various purposes, mainly in an effort to understand the dynamics of a system evolving through quantum measurements. In quantum dynamical systems, the ergodicity of a channel [3] and its decoherence-free subspaces [5] are important spectral properties. When the underlying domain is a bipartite system, the spectral properties of product channels can be hard to analyze, but we can use the multiplicative domain as a tool to understand them.

As in previous work on quantum error correction (e.g., [19][23]), we restrict our focus to unital channels because the multiplicative domain has less structure in non-unital channels. In particular, the multiplicative domain of a unital channel can be described using the Kraus operators. Using this, we can characterize certain channels and derive facts beyond the multiplicative structure.

The paper is organized as follows: firstly, in Section 2, we show that the multiplicative domain of a tensor product of unital channels "splits" nicely with the tensor product. We use this to prove that the peripheral spectra of two unital channels will precisely determine whether the fixed points of their tensor product will also split or not. This analysis provides the necessary and sufficient condition on when the tensor product of two ergodic (or primitive) channels is again ergodic (or primitive). Here we recapture some of the results obtained by [24], [32] in a very different way based on the analysis of multiplicative domain.

Since [28] showed that repeated applications of a finite-dimensional channel produces a chain in the lattice of unital *-subalgebras of $\mathcal{M}_d$, we characterize such algebras in Section 3.1. This provides an easy way to enumerate the lattice of unital *-subalgebras of $\mathcal{M}_d$, as well as providing a limit on the length of chains in the lattice that is linear in the dimension. This finding can be of independent interest because it provides a finer analysis of the structure of unital *-subalgebras in $\mathcal{M}_d$. In turn, this allows us to use the multiplicative index, introduced in [28], to show that certain channels cannot be product channels. We give examples of channels with large multiplicative indices in Sections 3.2 and 3.3, thus showing that these cannot be product channels.

Next, in Section 4, we consider channels which are strictly contractive with respect to some distinguishability measures that frequently arise in information theory. Using our results in the previous sections we prove that the tensor product of two strictly contractive channels with respect to certain distinguishability measures is again strictly contractive provided the measures allow recovery maps. We make use of the reversibility and monotonicity properties of these measures under channels, which is a wide topic of current research ([17], [18], [15], [14]).

As a final application, we show that unitary-correctable quantum codes (UCC) gain nothing through tensor products.

1.1. Background and notation

Throughout this paper we will use the following notation:

- $\mathcal{E}, \Psi$ will refer to quantum channels, that is, completely positive, trace-preserving linear operators from $B(\mathcal{H})$ to $B(\mathcal{H})$ for some finite dimensional Hilbert space $\mathcal{H}$. In this paper we identify $B(\mathcal{H})$ with $\mathcal{M}_d$, the