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Spreading speed and profiles of solutions to a free boundary problem with Dirichlet boundary conditions

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ABSTRACT

We discuss a free boundary problem for a reaction–diffusion equation with Dirichlet boundary conditions on both fixed and free boundaries of a one-dimensional interval. The problem was proposed by Du and Lin (2010) to model the spreading of an invasive or new species by putting Neumann boundary condition on the fixed boundary. Asymptotic properties of spreading solutions for such problems have been investigated in detail by Du and Lou (2015) and Du, Matsuzawa and Zhou (2014). The authors (2011) studied a free boundary problem with Dirichlet boundary condition. In this paper we will derive sharp asymptotic properties of spreading solutions to the free boundary problem in the Dirichlet case under general conditions on f. It will be shown that the spreading speed is asymptotically constant and determined by a semi-wave problem and that the solution converges to a semi-wave near the spreading front as $t \to \infty$ provided that the semi-wave problem has a unique solution.

1. Introduction

We discuss the following free boundary problem for a reaction–diffusion equation:

$$
\text{(FBP)} \quad \begin{cases} u_t - d u_{xx} = f(u), & t > 0,\ 0 < x < h(t), \\ u(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \le x \le h_0, \end{cases}
$$

where $\mu,\ h_0,\ d$ are given positive numbers, initial data (u_0, h_0) satisfies

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$$u_0 \in C[0,h_0] \cap C^2(0,h_0), \quad u_0(0) = u_0(h_0) = 0 \;\text{ and }\; u_0(x) > 0 \;\text{ in }\; (0,h_0) \tag{1.1}$$

and nonlinear function f meets

(A.1) $\quad f \in C^1[0,\infty),\ f(0) = f(1) = 0,\ f'(1) < 0,\ f(u) < 0$ for $u > 1$.

This type of a free boundary problem was proposed by Du–Lin [1] to model the spreading of an invasive or new species by putting Neumann condition on fixed boundary $x = 0$. See also Shigesada–Kawasaki [11] and Skellam [12] for the biological background of invasion models. After the appearance of [1], a lot of researchers have discussed such free boundary problems (see [2], [3], [4], [5], [6], [7], [8], [9] and the references therein). In particular, asymptotic properties of spreading solutions have been studied in great detail by Du–Lou [2] and Du–Matsuzawa–Zhou [3] in the case where both ends of the interval are moving boundaries determined by free boundary conditions of Stefan type.

In this paper we discuss the case where the habitat of a species is a one-dimensional interval and one end of the interval is a fixed boundary, whereas the other end is a moving boundary determined by Stefan condition $h'(t) = -\mu u_x(t, h(t))$. Homogeneous Dirichlet boundary conditions are imposed at the both ends of the interval. Biologically, this situation implies that the species cannot move across the fixed boundary and, therefore, it moves toward the moving boundary in order to get a new habitat. Denoting such an interval by $[0, h(t)]$ we consider the free boundary problem of the form (FBP). Under assumptions (1.1) and (A.1) we can show the following theorem on the existence and uniqueness of a global solution for (FBP).

Theorem 1. *Let f satisfy* (A.1). *Then for every initial data (u_0, h_0) satisfying* (1.1) *there exists a unique global solution (u, h) of* (FBP) *such that*

$$0 < u(t,x) \le C_1 \quad \textit{and} \quad 0 < h'(t) \le \mu C_2 \quad \textit{for} \;\; t > 0,\ 0 < x < h(t), \tag{1.2}$$

where C_1 and C_2 are positive constants depending only on $\|u_0\|_{C[0,h_0]}$ and $\|u_0\|_{C^1[0,h_0]}$, respectively. Moreover, it holds that

$$u_x(t,x) < 0 \tag{1.3}$$

for all (t,x) satisfying $t > 0$ and $\max\{h_0, h(t)/2\} \le x \le h(t)$.

One can refer to Kaneko–Yamada [6, Theorem 2.7] and Kaneko–Oeda–Yamada [5, Theorem 1.1] for the proof of Theorem 1 except for (1.3). For the proof of this decreasing property of u, see Lemma A.1 in Appendix.

Since the global existence of a unique solution to (FBP) has been established, the next stage is to study its asymptotic behavior as $t \to \infty$. It should be noted here that there are a pretty number of works for (FBP) and related problems. For instance, if f is of monostable type: f satisfies

$$\text{(A.1)} \quad \text{and} \quad f(u) > 0 \quad \text{for} \;\; u \in (0,1),$$

then it is well known that the spreading–vanishing dichotomy result holds true (see [6]). Hence any solution of (FBP) satisfies either spreading:

$$\lim_{t\to\infty} h(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} u(t,x) = v^*(x) \;\; \text{locally uniformly in } [0,\infty), \tag{1.4}$$

where $v^*(x)$ is a unique solution of