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Topological transitivity and mixing of composition operators

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ABSTRACT

Let $X = (X, \mathcal{B}, \mu)$ be a σ-finite measure space and $f : X \to X$ be a measurable transformation such that the composition operator $T_f : \varphi \mapsto \varphi \circ f$ is a bounded linear operator acting on $L^p(X, \mathcal{B}, \mu)$, $1 \le p < \infty$. We provide a necessary and sufficient condition on f for T_f to be topologically transitive or topologically mixing. We also characterize the topological dynamics of composition operators induced by weighted shifts, non-singular odometers and inner functions. The results provided in this article hold for composition operators acting on more general Banach spaces of functions.

1. Introduction

It is widely known that bounded linear operators acting on infinite dimensional Banach spaces may present chaotic behaviour (see [1,8]). Special attention has been devoted to the study of chaos in the sense of Devaney and Li-Yorke (see [2,3,7]). One of the key ingredients of chaos is the notion of topological transitivity. More specifically, a bounded linear operator T acting on a Banach space $\mathcal{X}$ is *topologically transitive* if for any pair $U, V \subset \mathcal{X}$ of nonempty open sets, there exists an integer $k \ge 0$ such that $T^k(U) \cap V \neq \emptyset$. When $\mathcal{X}$ is a separable Banach space, topological transitivity is equivalent to being *hypercyclic*, that is, to the existence of a dense T-orbit $\operatorname{Orb}(\varphi, T) = \{\varphi, T\varphi, T^2\varphi, \ldots\}$.

Here we are interested in the dynamics of the composition operator $T_f : \varphi \mapsto \varphi \circ f$ acting on a Banach space of functions $\mathcal{X}$. This is a subject that already appeared in the literature for regular functions, like holomorphic functions (see for instance [14]) or smooth functions (see [12]). Here, we focus on measurable functions and L^p-spaces. More precisely, let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and $(\mathcal{X}, \|\cdot\|) \subset L^0(\mu)$ be a Banach space of functions defined on X. We will always assume that $\mu(X) > 0$ and that $\mathcal{X}$ is a lattice: if

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ψ_1, ψ_2 are measurable functions with $|\psi_1| \le |\psi_2|$ and $\psi_2 \in \mathcal{X}$, then $\psi_1 \in \mathcal{X}$ and $\|\psi_1\| \le \|\psi_2\|$. We will say that $\mathcal{X}$ is *admissible* provided it satisfies the following assumptions

(H1) For any $A \in \mathcal{B}$ with finite measure, the function χ_A belongs to $\mathcal{X}$;
(H2) The set of simple functions which vanish outside a set of finite measure is dense in $\mathcal{X}$;
(H3) For all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $\psi \in \mathcal{X}$, for all $S \in \mathcal{B}$, $|\psi| \ge 1$ on S and $\|\psi\| \le \delta$ imply $\mu(S) < \epsilon/2$;
(H4) For all $\eta > 0$ and for all $M > 0$, there exists $\epsilon > 0$ such that for all $\psi \in \mathcal{X}$, for all $S \in \mathcal{B}$, $\mu(S) < \epsilon$, $\psi = 0$ on $X\backslash S$ and $|\psi| \le 2M$ imply $\|\psi\| < \eta/2$.

It is clear that L^p-spaces are admissible (in particular, (H3) follows from Markov's inequality). Throughout the paper, we will assume that $f : X \to X$ is nonsingular (namely $\mu\big(f^{-1}(S)\big) = 0$ whenever $\mu(S) = 0$). This ensures that $\varphi \circ f$ is well-defined for every $\varphi \in L^0(\mu)$. A necessary and sufficient condition for boundedness on $L^p(\mu)$ is the existence of some $c > 0$ such that $\mu\big(f^{-1}(B)\big) \le c\mu(B)$ for all $B \in \mathcal{B}$ (see [15]).

Our first result is a necessary and sufficient condition for the composition operator T_f to be topologically transitive. We do not make any extra assumption: neither X need to have finite measure nor f has to be bimeasurable or injective.

Theorem 1.1. *Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and $\mathcal{X}$ be an admissible Banach space of functions defined on X. Let $f : X \to X$ be measurable such that the composition operator $T_f : \varphi \mapsto \varphi \circ f$ is bounded on $\mathcal{X}$. Then the following assumptions are equivalent:*

(A1) *T_f is topologically transitive;*
(A2) *$f^{-1}(\mathcal{B}) = \mathcal{B}$ and, for all $\epsilon > 0$, for all $A \in \mathcal{B}$ with finite measure, there exist $B \subset A$ measurable, $k \ge 1$ and $C \in \mathcal{B}$ such that*

$$\mu(A\backslash B) < \epsilon,\ \mu(f^{-k}(B)) < \epsilon,\ f^k(B) \subset C \text{ and } \mu(C) < \epsilon.$$

We mention that a version of Theorem 1.1 has been already given by Kalmes in [9, Theorem 2.4] under additional assumptions. More specifically, he assumes that X is σ-compact, μ is locally finite and $f : X \to X$ is injective and continuous. As we show throughout this article, there are plenty of topologically transitive composition operators whose underlying map $f : X \to X$ is not continuous or is defined on a space X that it is not σ-compact.

Another key ingredient of chaos is the notion of topological mixing. A bounded linear operator $T : \mathcal{X} \to \mathcal{X}$ of a Banach space $\mathcal{X}$ is *topologically mixing* if for any pair $U, V \subset \mathcal{X}$ of nonempty open sets, there exists an integer $k_0 \ge 0$ such that $T^k(U) \cap V \ne \emptyset$ for every $k \ge k_0$. Our second result is the following.

Theorem 1.2. *Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and $\mathcal{X}$ be an admissible Banach space of functions defined on X. Let $f : X \to X$ be measurable such that the composition operator $T_f : \varphi \mapsto \varphi \circ f$ is bounded on $\mathcal{X}$. Then the following assumptions are equivalent:*

(B1) *T_f is topologically mixing;*
(B2) *$f^{-1}(\mathcal{B}) = \mathcal{B}$ and, for all $\epsilon > 0$, for all $A \in \mathcal{B}$ with finite measure, there exist $k_0 \ge 1$ and two sequences $(B_k), (C_k)$ of measurable subsets of X such that, for all $k \ge k_0$,*

$$\mu(A\backslash B_k) < \epsilon,\ \mu(f^{-k}(B_k)) < \epsilon,\ f^k(B_k) \subset C_k \text{ and } \mu(C_k) < \epsilon.$$

Conditions (A2) and (B2) are simplified when we add extra-assumptions on f or X. For instance, if f is one-to-one and bimeasurable, the condition $f^{-1}(\mathcal{B}) = \mathcal{B}$ is automatically satisfied and we do not need the