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Multiple positive and sign-changing solutions of an elliptic equation with fast increasing weight and critical growth

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A B S T R A C T

We consider the following equation

$$-\text{div}\big(K(x)\nabla u\big) = \lambda K(x)|x|^\beta |u|^{q-2}u + Q(x)K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $2 < q < 2^* = 2N/(N-2)$, $\lambda > 0$ is a parameter, $K(x) = \exp(|x|^\alpha/4)$, $\alpha \geq 2$, $\beta = (\alpha-2)\frac{(2^*-q)}{(2^*-2)}$ and $0 \leq Q(x) \in C(\mathbb{R}^N)$. Using variational methods and delicate estimates, we establish some existence and multiplicity of positive and sign-changing solutions for the problem, provided that the maximum of $Q(x)$ is achieved at different points.

1. Introduction

In this paper, we are concerned with the existence and multiplicity of positive and sign-changing solutions for the following problem:

$$-\text{div}\big(K(x)\nabla u\big) = \lambda K(x)|x|^\beta |u|^{q-2}u + Q(x)K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $2 < q < 2^* = 2N/(N-2)$, $\lambda > 0$ is a parameter, $K(x) = \exp(|x|^\alpha/4)$, $\alpha \geq 2$, $\beta = (\alpha-2)\frac{(2^*-q)}{(2^*-2)}$ and $0 \leq Q(x) \in C(\mathbb{R}^N)$ is assumed to satisfy the following condition:

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$(Q1)$ There exist k different points $a^1, a^2, \dots, a^k$ in $\mathbb{R}^N$ such that $Q(a^j)$ are strict maximums and satisfy

$$Q(a^j) = Q_M = \max\{Q(x) : x \in \mathbb{R}^N\} > 0, \quad j = 1, 2, \dots, k;$$

$(Q2)$ None of the points $a^1, a^2, \dots, a^k$ is an origin;

$(Q2')$ One of the points $a^1, a^2, \dots, a^k$ is an origin;

$(Q3)$ $Q_M - Q(x) = \begin{cases} o\big(|x - a^j|^{N-(N-2)q/2}\big), & \text{if } a^j \neq 0, \\ o\big(|x - a^j|^{N+\beta-(N-2)q/2}\big), & \text{if } a^j = 0, \end{cases}$ for x near a^j, $j = 1, 2, \dots, k$.

For $\alpha = q = 2, \lambda \equiv (N-2)/(N+2)$ and $Q(x) \equiv 1$, equation (1.1) is originated from finding self-similar solutions of the form

$$w(t, x) = t^{\frac{2-N}{N+2}} u(xt^{-1/2})$$

for the evolution equation

$$w_t - \Delta w = |w|^{4/(N-2)} w \quad on \quad (0, \infty) \times \mathbb{R}^N.$$

See [6,9] for a detailed description.

Equation (1.1) with $\alpha = q = 2, Q(x) \equiv 1$ has been treated by many authors. See [10,13,14,16] and reference therein. When $q = 2, Q(x) \equiv 1$, Catrina et al. [4] have obtained some existence results of the Brezis–Nirenberg type and have showed that the critical dimension of the problem depends on the value of α. Later on, when $Q(x) \equiv 1$, by using Mountain Pass Theorem and Linking Theorem, Furtado et al. [7] have proved that there are a positive solution if $2 < q < 2^*$ and a sign-changing solution if $q = 2$. Recently, Furtado et al. [8] have considered the following equation

$$-\mathrm{div}\big(K(x)\nabla u\big) = K(x) f(u) + \lambda K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $f(u)$ is superlinear and subcritical. In that article, for any given $k \in \mathbb{N}$, they have shown that there exists $\lambda^* = \lambda^*(k) > 0$ such that (1.2) has at least k pairs of solutions for $\lambda \in \big(0, \lambda^*(k)\big)$. But they can not give any information about the sign of these solutions. We also refer the interested reader to [2,3,15,20] for various existence results in the case $K(x) \equiv 1$ and $\alpha = 2$. As far as we know, we have not seen any multiplicity of positive and sign-changing solutions for problem (1.1) with the fast increasing weights $K(x)$ and $2 < q < 2^*$ in the literature.

The aim of this paper is to use the shape of the graph of $Q(x)$ to prove the existence and multiplicity of both positive and sign-changing solutions for problem (1.1), this property has been firstly observed by Cao and Noussair [2,3]. For the problem considered here, some different phenomena may appear since we have an additional weighted function $K(x)$. We will combine the effect of $K(x)$ and the shape of $Q(x)$ to study (1.1). Our main results are:

Theorem 1.1. *Assume conditions $(Q1)$, $(Q2)$ and $(Q3)$. If $N \geq 3$ and $\frac{2N-2}{N-2} < q < 2^*$, then there exists $\lambda_0 > 0$, such that (1.1) has at least k positive solutions for $\lambda \in (0, \lambda_0)$.*

Theorem 1.2. *Assume conditions $(Q1)$, $(Q2)$ and $(Q3)$. Then there exists $\lambda_0 > 0$, such that (1.1) has at least k sign-changing solutions, if one of the following statements holds:*

(i) $N \geq 4$, $\frac{2N-2}{N-2} < q < 2^*, \lambda \in (0, \lambda_0)$;

(ii) $N = 3, 5 < q < 2^*, \lambda \in (0, \lambda_0)$.

The following two Theorems consider a different case from the above two Theorems, in which one of the points $a^1, a^2, \dots, a^k$ is an origin.

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