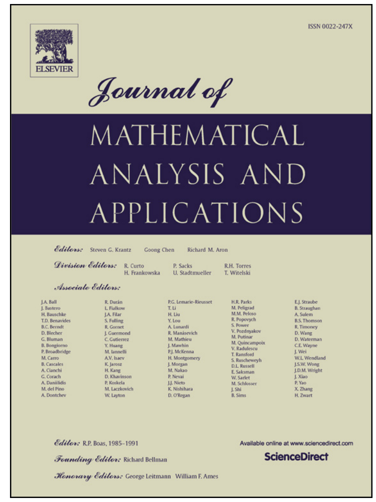

Some overdetermined problems related to the anisotropic capacity

Chiara Bianchini, Giulio Ciraolo, Paolo Salani

PII: S0022-247X(18)30383-4
DOI: https://doi.org/10.1016/j.jmaa.2018.04.075
Reference: YJMAA 22230

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 19 March 2018

SOME OVERDETERMINED PROBLEMS RELATED TO THE ANISOTROPIC CAPACITY

CHIARA BIANCHINI, GIULIO CIRAOLO, AND PAOLO SALANI

ABSTRACT. We characterize the Wulff shape of an anisotropic norm in terms of solutions to overdetermined problems for the Finsler p-capacity of a convex set $\Omega \subset \mathbb{R}^N$, with $1 < p < N$. In particular we show that if the Finsler p-capacitary potential u associated to Ω has two homothetic level sets then Ω is Wulff shape. Moreover, we show that the concavity exponent of u is $\mathsf{q} = -(p-1)/(N-p)$ if and only if Ω is Wulff shape.

AMS subject classifications. 35N25, 35B06, 35R25.
Key words. Wulff shape. Overdetermined problems. Capacity. Concavity exponent.

1. Introduction

The aim of this paper is to study some unconventional overdetermined problems for the Finsler p-capacity of a bounded convex set Ω associated to a norm H of $\mathbb{R}^N$, $N \geq 3$.

Given a bounded convex domain $\Omega \subset \mathbb{R}^N$, the p-capacity of Ω is defined by

$$\mathrm{Cap_p}(\Omega) = \inf \left\{ \frac{1}{p} \int_{\mathbb{R}^N} |D\varphi|^p \, dx, \ \varphi \in C_0^\infty(\mathbb{R}^N), \varphi(x) \geq 1 \text{ for } x \in \Omega \right\}$$

with $1 < p < N$. When the Euclidean norm $|\cdot|$ is replaced by a more general norm $H(\cdot)$, one can consider the so called *Finsler* p-capacity $\mathrm{Cap_{H,p}}(\Omega)$, which is defined by

$$\mathrm{Cap_{H,p}}(\Omega) = \inf \left\{ \frac{1}{p} \int_{\mathbb{R}^N} H^p(D\varphi) \, dx, \ \varphi \in C_0^\infty(\mathbb{R}^N), \varphi(x) \geq 1 \text{ for } x \in \Omega \right\}, \tag{1.1}$$

for $1 < p < N$. Under suitable assumptions on the norm H and on the set Ω, the above infimum is attained and

$$\mathrm{Cap_{H,p}}(\Omega) = \frac{1}{p} \int_{\mathbb{R}^N} H^p(Du_\Omega) \, dx,$$

where u_Ω is the solution of the Finsler p-capacity problem

$$\begin{cases} \Delta_p^H u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \\ u = 1 & \text{on } \partial\Omega, \\ u \to 0 & \text{as } H(x) \to +\infty. \end{cases} \tag{1.2}$$

Here Δ_p^H denotes the Finsler p-Laplace operator, i.e. $\Delta_p^H u = \mathrm{div}(H^{p-1}(Du)\nabla H(Du))$. The function u_Ω is named *(Finsler) p-capacitary potential* of Ω.

When Ω is Wulff shape, i.e. it is a sublevel set of the dual norm H_0

$$\Omega = B_{H_0}(r) = \{x \in \mathbb{R}^N : \ H_0(x - \bar{x}) < r\}$$

(see Section 2 for definitions), the solution to (1.2) can be explicitly computed and it is given by

$$v_r(x) = \left(\frac{H_0(x - \bar{x})}{r} \right)^{\frac{1}{\mathsf{q}}}, \tag{1.3}$$

with

$$\mathsf{q} = -\frac{p-1}{N-p}. \tag{1.4}$$

It is straightforward to verify that the potential v_r in (1.3) enjoys the following properties:

(i) the function v_r^{q} is convex, i.e. v_r is q-*concave*;
(ii) the superlevel sets of v_r are homothetic sets and they are Wulff shapes;
(iii) $H(Dv_r)$ is constant on the level sets of v_r.

The aim of this paper is to show that each of the properties (i)-(iii) characterizes the Wulff shape under some regularity assumptions on the norm H and on Ω. In particular, we assume that $H \in \mathcal{J}_p$ where

$$\mathcal{J}_p = \{H \in C_+^2(\mathbb{R}^N \setminus \{0\}), H^p \in C^{2,1}(\mathbb{R}^N \setminus \{0\}) \}. \tag{1.5}$$