J. Math. Anal. Appl. 465 (2018) 220–234

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

New criteria for the monotonicity of the ratio of two Abelian integrals

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ARTICLE INFO

Article history:
Received 9 October 2017
Available online 4 May 2018
Submitted by R. Popovych

Keywords:
Abelian integrals
Number of zeros
Planar systems

ABSTRACT

New criteria to determine the monotonicity of the ratio of two Abelian integrals are given. When two Abelian integrals have the forms $\int_{\Gamma_h} f_1(x)y\mathrm{d}x$ and $\int_{\Gamma_h} f_2(x)y\mathrm{d}x$ or the forms $\int_{\Gamma_h} \frac{f_1(x)}{y}\mathrm{d}x$ and $\int_{\Gamma_h} \frac{f_2(x)}{y}\mathrm{d}x$ and Γ_h are ovals belonging to the level set $\{(x,y)|H(x,y)=h\}$, where $H(x,y)$ has the form $y^2/2+\Psi(x)$ or $\phi(x)y^2/2+\Psi(x)$, we give new criteria, which are defined directly by the functions which appear in the above Abelian integrals, and prove that the monotonicity of the criteria implies the monotonicity of the ratios of the Abelian integrals. The new criteria are applicable in a large class of problems, some of which simplify the existing proofs and some of which generalize known results.

1. Introduction

This paper concerns the weakened 16th Hilbert problem on the number of limit cycles of plane differential systems, proposed by V. I. Arnold. The problem states as follows.

Consider a polynomial perturbation of a Hamiltonian system

$$\frac{dx}{dt} = \frac{\partial H(x,y)}{\partial y} + \varepsilon P(x,y), \quad \frac{dy}{dt} = -\frac{\partial H(x,y)}{\partial x} + \varepsilon Q(x,y),$$

where $H(x,y)$, $P(x,y)$ and $Q(x,y)$ are real polynomials. Then the Abelian integral associated to the above system is

$$I(h) = \int_{\Gamma_h} P(x,y)\mathrm{d}x - Q(x,y)\mathrm{d}y, \tag{1}$$

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https://doi.org/10.1016/j.jmaa.2018.04.074

along a closed level set $\Gamma_h \subset \{(x,y)|H(x,y) = h,\ h_1 < h < h_2\}$, where Γ_h forms a continuous family of ovals as h varies in the open interval (h_1, h_2). The question one asks is: how large can the number of isolated zeros of the function $I(h)$ be in the above open interval when P, Q and H are polynomials whose degrees are known? This problem is related to the estimation of the number of limit cycles of the perturbed Hamiltonian system. On this theme there have been many excellent works, for example, Binyamini et al. in [4] obtained a double exponential upper bound in n on the number of zeros of Abelian integral where $\deg H = n + 1$ and $\deg P, Q = n$. For more works, we recommend the readers the review papers [14,15] or the book [6].

In this paper we consider the case where

$$H(x,y) = \frac{y^2}{2} + \Psi(x), \tag{2}$$

with $\Psi(x) \in C^2(\mu, \nu)$, $\mu, \nu \in \mathbb{R}$. In this case, the Abelian integral (1) can be written as

$$\sum_{k=1}^{m} \alpha_k I_k(h),$$

where $\alpha_1, \alpha_2, \cdots, \alpha_m$ are real constants, and the I_k are in the form

$$I_k(h) = \int\limits_{\Gamma_h} f_k(x) y \mathrm{d}x, \qquad k = 1, 2, \cdots, m,$$

where $f_k(x)$ are functions of class C^1.

Assume that one of the first two integrals $I_1(h)$ and $I_2(h)$ is non-vanishing, without loss of generality, that is $I_1(h)$. We let

$$u(h) = \frac{I_2(h)}{I_1(h)}.$$

Then the monotonicity of the ratio $u(h)$ shows that the Abelian integral (1) has at most one zero if $m = 2$. If $m \geq 2$, as a first step, the monotonicity of the ratio $u(h)$ also play an important role in determining the number of zeros of the Abelian integral (1). For more details, see [16].

There have been many methods to obtain the monotonicity of the ratio $u(h)$, for example:

(i) using Picard Fuchs equations, see [19,7];
(ii) using Green formula, see [5];
(iii) direct estimation of the integral $I_2(h)I_1'(h) - I_1(h)I_2'(h)$ by using some technical tools, see [18,21,17].

Each of the above methods can be only used to some special cases, and one has to repeat the whole procedure of calculations for each individual problem. In [16], Li and Zhang develop a direct method. Concretely, they give a criterion function $\xi(x)$ depending only on $f_1(x), f_2(x), \Psi(x)$ and prove that the monotonicity of $\xi(x)$ implies the monotonicity of $u(h)$. Let us revisit their result.

Consider the Hamiltonian system

$$\dot{x} = y, \qquad \dot{y} = -\Psi'(x), \tag{3}$$

which has the Hamiltonian function $H(x,y)$ in form (2).

Assume that there exists an number $a \in (\mu, \nu)$ such that the following hypothesis is satisfied:

$$(H1) \qquad \Psi'(x)(x-a) > 0, \quad \text{for all } x \in (\mu,\nu)\backslash\{a\}.$$