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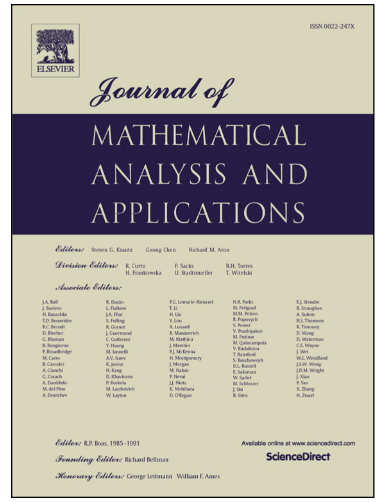

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P. Del Moral, A. Niclas

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A Taylor expansion of the square root matrix function

P. Del Moral[*], A. Niclas[†]

Abstract

This short note provides an explicit description of the Fréchet derivatives of the principal square root matrix function at any order. We present an original formulation that allows to compute sequentially the Fréchet derivatives of the matrix square root at any order starting from the first order derivative. A Taylor expansion at any order with an integral remainder term is also provided, yielding the first result of this type for this class of matrix function.

Keywords : Fréchet derivative, square root matrices, Taylor expansion, Sylvester equation, spectral and Frobenius norms, matrix exponential.

Mathematics Subject Classification : 15A60, 15B48, 15A24.

1 Introduction

The computation of matrix square roots arise in a variety of application domains, including in physics, signal processing, optimal control theory, and many others. The literature abounds with numerical techniques for computing matrix square roots, see for instance [1, 12, 13, 17, 18, 19, 22]. Perturbative techniques often resume to Lipschitz type estimates [4] or on the refined analysis of the first order Fréchet derivative of the principal square root matrix function; see for instance [1, 2, 3, 24], as well as chapter X in the seminal book by R. Bhatia [5] and references therein. We also refer to the article [10] for a first order analysis of more general matrix n-th roots. For further details on the n-th roots of matrices we refer to [23].

The purpose of this article is to derive an explicit description of the Fréchet derivatives of the principal square root matrix function at any order. We also provide a non asymptotic Taylor expansion at any order, with computable estimates of the integral remainder terms. These expansions provide a perturbation computation of the square root $\sqrt{A+H}$ of a positive definite matrix A perturbed by some symmetric matrix H, as soon as $A + \epsilon\ H$ is positive semidefinite for any $\epsilon \in [0, 1]$.

We underline that the perturbation analysis developed in this article differs from Taylor expansion type techniques often used to define functions on the spectrum of diagonalizable matrices via Jordan canonical forms. This Sylvester's formulation of matrices are related to the Sylvester matrix theorem (a.k.a. Lagrange-Sylvester interpolation) which allows to express an analytic function of a matrix in terms of its eigenvalues and eigenvectors. For a more thorough discussion on these interpolation techniques we refer to the first chapter in the seminal book by N. J. Higham [16].

[*]INRIA Bordeaux Sud Ouest Center, 33 405 Talence, pierre.del-moral@inria.fr

[†]ENS Lyon, 46 Allée d'Italie, 69007 Lyon, angele.niclas@ens-lyon.fr