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Elliptic 1-Laplacian equations with dynamical boundary conditions

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ABSTRACT

This paper is concerned with an evolution problem having an elliptic equation involving the 1-Laplacian operator and a dynamical boundary condition. We apply nonlinear semigroup theory to obtain existence and uniqueness results as well as a comparison principle. Our main theorem shows that the solution we found is actually a strong solution. We also compare solutions with different data.

1. Introduction

In this paper we deal with existence and uniqueness for an evolution problem. It consists in an elliptic equation involving the 1-Laplacian operator and a dynamical boundary condition, namely,

$$\begin{cases} \lambda u - \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = 0\,, & \text{in } (0,+\infty)\times\Omega\,, \\ \omega_t + \left[\dfrac{Du}{|Du|}, \nu\right] = g(t,x)\,, & \text{on } (0,+\infty)\times\partial\Omega\,, \\ u = \omega\,, & \text{on } (0,+\infty)\times\partial\Omega\,, \\ \omega(0,x) = \omega_0(x)\,, & \text{on } \partial\Omega\,; \end{cases} \tag{1}$$

where Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\lambda > 0$, ν stands for the unit outward normal vector on $\partial\Omega$, $g \in L^1_{loc}(0,+\infty; L^2(\partial\Omega))$ and $\omega_0 \in L^2(\partial\Omega)$. Here, we have denoted by ω_t the distributional derivative of ω with respect to t. As far as we know, this is the first time that dynamical boundary conditions for the 1-Laplacian are considered.

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We point out that dynamical boundary conditions naturally occur in applications where there is a reaction term in the problem that concentrates in a small strip around the boundary of the domain, while in the interior there is no reaction and only diffusion matters. So, it appears in many mathematical models including heat transfer in a solid in contact with a moving fluid, in thermoelasticity, in biology, etc. This fact has given rise to many papers (see [1,5,6,8,12–14,17,21,23,24,30]) dealing with problems having dynamical boundary conditions, and mainly of those problems involving linear operators. The study of problems where an elliptic or parabolic equation occurs with this kind of boundary conditions is nowadays an active branch of research and we refer to [18,20,22,28,31] and references therein for recent papers.

The study of an evolution problem having an elliptic equation driven by the p-Laplacian (with $p > 1$) and a dynamical boundary condition is due to [5] (see also [6]). To handle with that nonlinear problem, the authors define a completely accretive operator, apply the nonlinear semigroup theory to get a mild solution and finally, prove that this mild solution is actually a weak solution. Once their result is available, we may study problem (1) taking the solution corresponding to $p > 1$ and letting p go to 1. Nevertheless, we are not able to pass to the limit and this approach remains an open problem. Furthermore, once a solution to our problem is obtained, we cannot prove that it is the limit of mild solutions to problems involving the p-Laplacian. What we need to prove the convergence would be a Modica type result on lower semicontinuity (see [29, Proposition 1.2]) for functionals depending on time.

Instead trying this approach, we adapt the method used in [5] and apply the nonlinear semigroup theory (we refer to [9] for a good introduction to this theory). Obviously, the singular features of the 1-Laplacian do not allow us to follow every step. Among the special features verified by the 1-Laplacian, we highlight that boundary conditions need not be satisfied in the sense of traces (we refer to [4] for the Dirichlet problem, to [27] for the Neumann problem as well as [3] for the homogeneous Neumann for a related equation, and to [26] for the Robin problem). This fact leads us to modify the procedure from the very beginning since it implies a change in the definition of the associated accretive operator. Indeed, the translation of the operator studied in [5] to our setting would be an operator $\mathfrak{B} \subset L^2(\partial\Omega) \times L^2(\partial\Omega)$ defined as follows:

Definition 1.1. Let $v, \omega \in L^2(\partial\Omega)$. Then $v \in \mathfrak{B}(\omega)$ if there exists $u \in BV(\Omega) \cap L^2(\Omega) \cap L^2(\partial\Omega)$ such that $u\big|_{\partial\Omega} = \omega$ and it is a solution to the Neumann problem

$$\begin{cases} \lambda u - \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = 0\,, & \text{in } \Omega\,; \\ \left[\dfrac{Du}{|Du|} \cdot \nu\right] = v\,, & \text{on } \partial\Omega\,. \end{cases}$$

This is indeed a completely accretive operator but, unfortunately, we are not able to prove that it satisfies the range condition; thus the nonlinear semigroup theory cannot be applied. We turn out to define our operator for $v, \omega \in L^2(\partial\Omega)$ as $v \in \mathcal{B}(\omega)$ if $v \in L^\infty(\partial\Omega)$, with $\|v\|_{L^\infty(\partial\Omega)} \le 1$, and there exists $u \in BV(\Omega) \cap L^2(\Omega)$ which is a solution to the Dirichlet problem with datum ω and it is also a solution of the Neumann problem with datum v (see Definition 3.2 below). Now, we do not know if this operator is completely accretive, we only prove that it is accretive in $L^2(\partial\Omega)$. Hence, we have not to expect that our solution holds every feature satisfied by solutions to problems driven by the p-Laplacian (for instance, we just choose initial data belonging to $L^2(\partial\Omega)$). Moreover, even when our solution satisfies the same property, the proof of this fact can be different, as can be checked in the comparison principle. Despite these difficulties, we obtain global existence and uniqueness of solution for every datum $\omega_0 \in L^2(\partial\Omega)$ as well as a comparison principle. Furthermore, we prove that the solution we found is a strong solution in the sense that the problem holds for almost all $t > 0$. We also analyze some related properties as the continuous dependence on data. Our main result is the following.