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A construction of two different solutions to an elliptic system

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A B S T R A C T

The paper aims at constructing two different solutions to an elliptic system

$$u \cdot \nabla u + (-\Delta)^m u = \lambda F$$

defined on the two dimensional torus. It can be viewed as an elliptic regularization of the stationary Burgers 2D system. A motivation to consider the above system comes from an examination of unusual properties of the linear operator $\lambda \sin y \partial_x w + (-\Delta)^m w$ arising from a linearization of the equation about the dominant part of F. We argue that the skew-symmetric part of the operator provides in some sense a smallness of norms of the linear operator inverse. Our analytical proof is valid for a particular force F and for $\lambda > \lambda_0$, $m > m_0$ sufficiently large. The main steps of the proof concern finite dimension approximation of the system and concentrate on analysis of features of large matrices, which resembles standard numerical analysis. Our analytical results are illustrated by numerical simulations.

1. Introduction

The analysis of sets of solutions to elliptic systems/equations is of particular interest in the ongoing research on partial differential equations (PDEs). On the one hand, the question is challenging from the viewpoint of mathematical techniques. On the other hand, precise information about this set is crucial for understanding the dynamics of evolutionary problems behind the elliptic one. In general, existing theory provides us with two answers; either there exists a single solution or the system admits at least one solution with unknown multiplicity.

Existing methods of PDEs analysis provide only few example proofs of existence of multiple solutions for quite simple problems, like the classical Mountain Pass Theorem for a semilinear elliptic equation [17]. Other notable examples are: nonuniqueness of stationary solutions to the Navier–Stokes equations [18], geometric

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results related to the mean curvature problems [9], and nonuniqueness of solutions for the one-dimensional viscous Burgers' equation [4] or the evolutionary Burgers' equation [16], [1]. A derivation of an asymptotic lower bound for the multiplicity of solutions of a semilinear problem can be found in [15], [22], and for a class of elliptic equations with jumping nonlinearities in [24]. Let us mention a related, but having a different flavor than the mentioned results, research on numerical multiplicity proofs for systems/higher dimensional PDEs. There exist several computer assisted proofs of existence of at least several solutions for certain parabolic PDEs. Let us emphasize that contrary to our approach, all results obtained using some direct computer assistance must hold essentially for some isolated parameter values or a compact set of parameter values, because computations performed by any digital computer are finite. Some representative results include: a proof of existence of four solutions to a semilinear boundary value problem [8], a proof of existence of nonsymmetric solutions to a symmetric boundary value problem [2], validated bifurcation diagrams [7], [19], structure of the global attractor [13,23], numerical existence proofs for a fluid flow, and convection problems [27], [20].

The subject studied in the present paper is the following elliptic system, which can be viewed as an elliptic regularization of the *stationary Burgers system* [10], [21], [11] *in 2D*

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} + (-\Delta)^m \boldsymbol{u} = \lambda \boldsymbol{F} \text{ on } \mathbb{T}^2. \tag{1}$$

Here $\boldsymbol{u}$ is sought as a vector function $\boldsymbol{u} : \mathbb{T}^2 \to \mathbb{R}^2$. The vector $\boldsymbol{F}$ is an external force, and in this paper we define it as

$$\boldsymbol{F}(x, y) = \begin{pmatrix} \sin y \\ \sin x \end{pmatrix}. \tag{2}$$

The magnitude of the external force is controlled using parameter λ and it is assumed to be greater than some positive number λ_0. We shall note that the system has *no a-priori estimate*. The issue of the existence of a solution to the system (1) is still open for a general form of $\lambda \boldsymbol{F}$. To the best of our knowledge even the basic case of $m = 1$ is unclear.

Let us discuss what motivated the presented research. Our numerical investigations of (1) revealed a solution possessing a peculiar structure: one Fourier mode having λ magnitude, and the remainder bounded uniformly with respect to λ. We noticed further that the natural symmetry embedded in this equation implies the existence of a second solution, as the reflection of the dominant part produces a different symmetric solution. Further on, to convince ourselves that this structure is in fact conserved for λ large, we performed a numerical bifurcation analysis, which showed that the solution's norm graph is approximately linear, and in fact the system admits an apparent pitchfork bifurcation.

We emphasize that the studied solutions are not trivial. For λ sufficiently small the solutions are still symmetric, and for some particular λ the symmetry gets broken, which allows for establishing the existence of at least two distinct solutions for λ sufficiently large. Nonetheless, for small λ regime we can claim only existence of a solution, as the two solutions from our main result merge into a single one there. The symmetry is elementary; it swaps x with y, and the first component of the solution with the second (denoted by $x \leftrightarrow y$ in the sequel). Apparently, a stronger diffusive regularization effect than the one provided by Laplacian is required for our method to work. This is why we state our main result (Theorem 1.1) for m sufficiently large. Our analytical results are supported by a numerical bifurcation analysis (Section 3).

The main tool of our technique is to exploit unusual features of a linearization of the system. Let $\|w\|_{l^\infty}$ denote the supremum norm of elements of the Fourier series w. Apparently, for the solutions to the following scalar problem

$$\lambda \sin y \partial_x w + (-\Delta)^m w = \lambda \sin x \text{ on } \mathbb{T}^2, \tag{3}$$