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Perturbation of the tangential slit by conformal maps ☆

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ABSTRACT

For a tangential slit, the behavior of the driving function in the Loewner differential equation is less clear. In this paper, we investigate the tangential slit $\varphi(\Gamma)$, where φ is a univalent real analytic function near the origin, and where Γ is a circular arc tangent at the origin. Our main aim is to give an interesting way to prove the asymptotic property of the driving function which generates the tangential slit $\varphi(\Gamma)$.
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1. Introduction

The Loewner differential equation was introduced by Loewner in 1923 to study the Bieberbach conjecture [1], and it led to a proof of the case $n = 3$ [3]. When the conjecture finally was proved by de Brange [2], the Loewner equation again played a key role. In 2000, Schramm [18] found a description of the scaling limits of some stochastic processes in terms of the Loewner equation. This led to the definition of a new stochastic process, the so-called Stochastic Loewner Evolution (SLE). Many results in this fast-growing field can be found in the recent work of mathematicians such as Lawler, Rohde, Schramm, Werner, Smirnov ([11,16,18,19], [10] and the references there). This also re-ignited the interest of the Loewner equation and its solution in the deterministic case ([4,7,9,12–15,20,21]).

Let $\mathbb{H}$ be the upper half-plane. Suppose for any $T > 0$, $\gamma : [0, T] \to \overline{\mathbb{H}}$ is a simple curve with $\gamma(0) \in \mathbb{R}$ and $\gamma(0, T] \subset \mathbb{H}$. For every $t \in [0, T]$, the region $H_t = \mathbb{H} \setminus \gamma[0, t]$ is simply connected. Then there is a unique conformal map g_t from H_t onto $\mathbb{H}$ with the hydrodynamic normalization (see Part 2):

$$g_t(z) = z + \frac{c(t)}{z} + O(\frac{1}{z^2}), \quad \text{as } z \to \infty.$$

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It is easy to check that $c(t)$ is continuously increasing in t and $c(0) = 0$. Therefore γ can be reparametrized so that $c(t) = 2t$, and this γ is said to be *parameterized by the half-plane capacity*. In this case, $g_t(z)$ satisfies the equation

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z, \tag{1.1}$$

where $\lambda(t) := \lim_{z\to\gamma(t)} g_t(z)$ is a continuous real-valued function. The equation (1.1) is called the *(chordal) Loewner (differential) equation*, λ is called the *driving function* or the *Loewner transform*, and γ is called the *trace* or the *Loewner curve*.

On the other hand, given a continuous function $\lambda : [0, T] \to \mathbb{R}$ and $z \in \mathbb{H}$, we can solve the initial value problem (1.1). Let T_z be the supremum of all t such that the solution is well defined up to time t with $g_t(z) \in \mathbb{H}$. Let $H_t = \{z \in \mathbb{H} : T_z > t\}$. Then one can prove that the set H_t is a simply connected subdomain of $\mathbb{H}$ and g_t is the unique conformal transformation from H_t onto $\mathbb{H}$ with the hydrodynamic normalization:

$$g_t(z) = z + \frac{2t}{z} + O(\frac{1}{z^2}), \quad \text{as } z \to \infty.$$

Let $K_t = \mathbb{H} \setminus H_t$. Then $\{K_t\}_{t\in[0,T]}$ is an increasing family of hulls (see Part 2) in $\mathbb{H}$, and we say that the hulls K_t are generated by the driving function λ. In general, it is not true that the hulls K_t generated by a continuous driving function λ is a *slit*, i.e., $K_t = \gamma(0, t]$ for some simple curve γ with $\gamma(0) \in \mathbb{R}$ and $\gamma(0, T] \subset \mathbb{H}$.

For non-tangential slits, many results are given around Lip(1/2), which is the space of Hölder continuous functions with exponent 1/2. Marshall, Rohde [14] and Lind [12] proved that H_t is a slit half-plane for all t provided that $\|\lambda\|_{1/2} < 4$; conversely, $\lambda \in \text{Lip}(1/2)$ if γ is a quasiarc that approaches $\mathbb{R}$ non-tangentially. Wong [20] gave the proof of the smoothness of the trace. For tangential slits, recently Prokhorov and Vasil'ev [15] showed that the circular arc

$$\Gamma := \{i + e^{i\theta} : -\frac{\pi}{2} \le \theta \le 0\} \tag{1.2}$$

is generated by a Hölder continuous driving function with exponent 1/3; Lau and Wu [9] considered the tangential slits $\Gamma^p (p > 0)$, and gave the exact expression of the driving functions.

In this paper, we consider the behavior of the driving function which generates the tangential slit $\varphi(\Gamma)$, where Γ is defined in (1.2), and where φ is an analytic function defined in the neighbourhood of the origin of the form

$$\varphi(w) = a_1 w + a_2 w^2 + a_3 w^3 + a_4 w^4 + \cdots, \qquad a_1 > 0,\ a_n \in \mathbb{R}. \tag{1.3}$$

Our main theorem is

Theorem 1.1. *Let $\varphi(\Gamma)$ be the trace generated by the Loewner equation (1.1). Then its driving function λ is of the form*

$$\lambda(t) = (12\pi\varphi'(0)t)^{\frac{1}{3}} + o(t^{\frac{1}{3}}), \quad \text{as } t \to 0.$$

In order to prove Theorem 1.1, some basic results we will need are given in Part 2. Then the main proof of this theorem will be divided into the following two parts: Part 3 and Part 4. In Part 3, we will consider the special case that φ^{-1} has the form $z(1 - c_1 z)(1 - c_2 z)\cdots(1 - c_n z)$. We can obtain an integral expression P_t using the Christoffel–Schwarz formula which is applied in the Riemannian manifold ([5,6]), and therefore we can get three equalities about the driving function $\lambda(t)$ via the function P_t. Then we obtain the asymptotic

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