J. Math. Anal. Appl. 465 (2018) 588–603

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The countable sup property for lattices of continuous functions ☆

M. Kandić, A. Vavpetič *

Fakulteta za Matematiko in Fiziko, Univerza v Ljubljani, Jadranska ulica 19, SI-1000 Ljubljana, Slovenia

ARTICLE INFO

Article history:
Received 13 June 2017
Available online 9 May 2018
Submitted by M. Mathieu

Keywords:
Vector lattices
Continuous functions
Countable sup property
Chain conditions
Strictly positive functionals

ABSTRACT

In this paper we find sufficient and necessary conditions under which vector lattice $C(X)$ and its sublattices $C_b(X)$, $C_0(X)$ and $C_c(X)$ have the countable sup property. It turns out that the countable sup property is tightly connected to the countable chain condition of the underlying topological space X. We also consider the countable sup property of $C(X \times Y)$. Even when both $C(X)$ and $C(Y)$ have the countable sup property it is possible that $C(X \times Y)$ fails to have it. For this construction one needs to assume the continuum hypothesis. In general, we present a positive result in this direction and also address the question when $C(\prod_{\lambda \in \Lambda} X_\lambda)$ has the countable sup property. Our results can be understood as vector lattice theoretical versions of results regarding products of spaces satisfying the countable chain condition. We also present new results for general vector lattices that are of an independent interest.

1. Introduction

In topology, when one deals with continuous functions, there are two possibilities. One can work either by open sets or by nets (generalized sequences). Although dealing with nets is maybe computationally easier, one needs to be cautious since there is a variety of different types of nets and one can often make errors. When the topology of a given space is metrizable (more general first-countable), the sequential nature of the space enables us to work by sequences instead of nets. Since order convergence in vector lattices is also defined through nets, one would also like to pass from nets to sequences, of course, if possible. In the setting of vector lattices this notion is called the ***countable sup property***. It plays an important role in the recent research in vector and Banach lattices. For example, in [1] it was used to prove that every function in $C(\mathbb{R}^m)$ is the order limit of an order convergent sequence of piecewise affine functions. Next, in [10] authors used it to prove that in some Banach function spaces over σ-finite measure spaces convex Komlós sets are norm

☆ The first author acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1-0222). The second author acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1-0292, J1-8131 and J1-7025).

* Corresponding author.
E-mail addresses: marko.kandic@fmf.uni-lj.si (M. Kandić), ales.vavpetic@fmf.uni-lj.si (A. Vavpetič).

https://doi.org/10.1016/j.jmaa.2018.05.028
0022-247X/

bounded. Last but not least, in [16] authors proved that the universal completion of a vector lattice with a weak unit and with the countable sup property also has the countable sup property. With this result they proved uo-completeness of the universal completion of some vector lattices (see [16, Theorem 2.10]). Recall that a vector lattice E is said to be uo-complete whenever every uo-Cauchy net in E uo-converges in E.

These days uo-convergence plays a very important role in the research of vector lattices. Although uo-convergence is very exciting on its own, its value shows through its applications in Mathematical finance. In [12], the countable sup property served as an indispensable tool in the study of option spanning of the option space O_f of a limited liability claim f within some order ideal X in the vector lattice $L_0(\mathbb{P})$ of all measurable functions defined on a probability space $(\Omega, \Sigma, \mathbb{P})$. The key ingredient is the characterization of order closed sublattices of X which contain constant functions (see [12, Lemma 2.1] and [12, Lemma 2.2]). The results on option spanning were used to prove that the price of any contingent claim $g \in L_0(\sigma(f)) \cap X$ is determined by an arbitrage from (O_f, π) under the assumption that (O_f, π) admits no free lunches where π is a positive functional on O_f and $\sigma(f)$ is the smallest σ-subalgebra in Σ under which f is measurable. The results of [12] have led to the recent development (see [7]) on the question "what is the smallest order closed vector sublattice which contains a given vector sublattice".

For general results on uo-convergence and its unbounded norm version we refer the reader to [11,6,14,10,16]. For applications of uo-convergence and its techniques to Mathematical finance we refer the reader to [12,7–9].

In this paper our interest is the countable sup property itself. Although we present some new general results which also extend some results from [1] and [16], our main concern is the countable sup property for vector lattices of continuous functions on a given topological space.

The paper is structured as follows. In Section 2 we introduce notation and basic notions needed throughout the text. In Section 3 we introduce different chain conditions on a topological space X and prove that the existence of a strictly positive functional on $C_b(X)$ implies the weakest of them. In Section 4 we connect the countable sup property of $C(X)$ to chain conditions from Section 3. It turns out that the countable chain condition of a topological space X implies that $C(X)$ has the countable sup property and that, in general, they are not equivalent. However, they are equivalent when $X \in T_{3\frac{1}{2}}$. Also, for a metric space X, the countable sup property of $C(X)$ is equivalent to separability of X. Along the way we extend two results from [1] and [16]. In Section 5 we prove that vector lattices $C_c(X)$ and $C_0(X)$ simultaneously have the countable sup property or simultaneously fail to have it. In the last section we consider the vector lattice $C(X \times Y)$. It is possible for both $C(X)$ and $C(Y)$ to have the countable sup property while $C(X \times Y)$ lacks it. This follows under continuum hypothesis from Galvin's example [5] and Proposition 4.8. This example also leads to an example of an extremally disconnected compact Hausdorff space X such that $C(X)$ has the countable sup property while $C(X \times X)$ lacks it (see e.g. [20]). At this place we would like to mention that in the case when $X \in T_{3\frac{1}{2}}$ the vector lattice $C(X)$ is order complete iff X is extremally disconnected (see [2, Theorem 1.50]). We also prove that whenever $C(X)$ has the countable sup property and Y is separable, then $C(X \times Y)$ has the countable sup property. This result can be considered as a vector lattice version of [28, Theorem 3.3]. Last but not least, we also prove that $C(\prod_{\lambda \in \Lambda} X_\lambda)$ has the countable sup property whenever for each finite family $\Lambda_0 \subseteq \Lambda$ the space $C(\prod_{\lambda \in \Lambda_0} X_\lambda)$ has the countable sup property. Again, this can be considered as a vector lattice version of [20, Theorem 2.2].

2. Preliminaries

Throughout the paper, if not otherwise stated, vector lattices are assumed to be Archimedean. A vector x of a vector lattice E is said to be ***positive*** if $x \geq 0$. The set of all positive vectors of E is denoted by E_+. A vector $e \in E_+$ is said to be a ***unit*** if for every $x \in E$ there is some $\lambda \geq 0$ such that $|x| \leq \lambda e$. A positive vector $e \in E$ is said to be a ***weak unit*** if $|x| \wedge e = 0$ implies $x = 0$. A vector sublattice F is ***order dense*** in E if for each nonzero $x \in E_+$ there is $y \in F_+$ satisfying $0 < y \leq x$. When E is Archimedean, F is