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Note

# A remark on the probabilistic solution of the Dirichlet problem for simply connected domains in the plane

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ABSTRACT

A new proof is given of a simple probabilistic lemma which implies the solution of the Dirichlet problem for simply connected domains in the plane. This proof uses the conformal invariance of planar Brownian motion in place of the previously existing measure-theoretic arguments.



Consider the following formulation of the Dirichlet problem in $\mathbb{R}^n$.

**Dirichlet problem.** *If $U$ is a bounded region, and $h$ a continuous function on $\delta U$, to extend $h$ to a harmonic function on $U$ which is continuous on the closure of $U$.*

It is well known that in many cases the Dirichlet problem may be solved by running a Brownian motion. To be precise, we may let $h(a) = E_a[h(B_{T(U)})]$, where $T(U) = \inf\{t > 0 : B_t \in U^c\}$; the Strong Markov Property of Brownian motion shows easily that $h$ is harmonic, but the difficulty lies in showing that $h$ is continuous on the closure of $U$, if it is true (there are domains, such as a punctured disk, for which the Dirichlet problem is not solvable). Of particular interest to complex analysts is the case of a simply connected domain in the plane, and we have the following classical result.

**Theorem 1.** *The Dirichlet problem is solvable for any simply connected domain in $\mathbb{C}$.*

A proof of this result using Brownian motion exists, and is presented in [1, Prop. II.1.14 and Thm. II.1.15], with variations or partial results with the same theme being presented in other sources, such as [2] and [3]. However, this proof uses a number of rather sophisticated probabilistic results (such as Theorem 2 below) and is therefore likely to be inaccessible to most analysts. The purpose of this note is to present a short

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"proof by picture" of this result; the proof depends on the conformal invariance of Brownian motion and we hope it will be readily understood by complex analysts.

Let us begin with a few words on the proof of Theorem 1 given in [1], so that the context of the result may be seen. Let $B_t$ denote a planar Brownian motion and $U$ a simply connected domain. A boundary point $y$ of $U$ is called *regular* if $P_y(T(U) = 0) = 1$. It is then proved that any boundary point in a simply connected domain is regular ([1, Prop. II.1.14]) by combining the Blumenthal Zero–One Law ([1, Cor. I.3.6]) with a path constructed by the following support theorem ([1, Thm. I.6.6]):

**Theorem 2.** *If $\psi : [0, t] \longrightarrow \mathbb{C}$ is continuous, $\varepsilon > 0$, and $\psi(0) = a$, then*

$$P_a(\sup_{0\leq s\leq t} |B_s - \psi(s)| < \varepsilon) > c,$$

*where $c$ can be taken to depend only on $t, \varepsilon$, and the modulus of continuity of $\psi$.*

This powerful theorem is far from simple, and its proof depends on a Girsanov change of Wiener measure. Several other manipulations ([1, Prop. II.1.10, Cor. II.1.11, Prop. II.1.12]) then show that a Brownian motion started near a regular boundary point is likely to exit near that point (essentially Lemma 1 below), and the result follows from this by writing

$$E_a[h(B_{T(U)})] = E_a[h(B_{T(U)})1_{\{|B_{T(U)}-y|<\varepsilon\}}] + E_a[h(B_{T(U)})1_{\{|B_{T(U)}-y|\geq\varepsilon\}}]$$

and noting that for $a$ sufficiently close to $y$ the first term on the right is as close as we like to $h(y)$ and the second can be made as small as we like (using the boundedness of $h$).

This approach does have its advantages, in particular it generalizes well to regions in higher dimensions; however, for simply connected domains in the plane the relationship between Brownian motion and analytic functions provides us with an alternate path which may appeal to analysts. In fact, we may completely bypass the support theorem, and can even disregard the concept of regularity being defined as above in terms of time, using instead Lévy's theorem on conformal invariance and a well chosen stopping time. We will prove the following, which clearly suffices for our purposes.

**Lemma 1.** *If $y$ is a boundary point of a simply connected domain $U$, then given any $\varepsilon > 0$ we can find $\delta > 0$ such that $P_a(|B_{T(U)} - y| < \varepsilon) > 1 - \varepsilon$ whenever $|a - y| < \delta$.*

As indicated above this implies Theorem 1 immediately. To prove this, we will need to make use of Lévy's Theorem on conformal invariance ([1, Thm. V.1.2]):

**Theorem 3.** *Let $f$ be analytic and nonconstant on a domain $U$, and let $a \in U$. Let $B_t$ be a planar Brownian motion which starts at $a$ and is stopped at a stopping time $\tau \leq T(U)$. Set*

$$\sigma(t) = \int_0^{t\wedge\tau} |f'(B_s)|^2 ds.$$

*$\sigma(t)$ is a.s. strictly increasing and continuous on $[0, \tau]$, and we let $C_t = \sigma^{-1}(t)$ on $[0, \sigma(\tau)]$. Then $\hat{B}_t = f(B_{C_t})$ is a Brownian motion stopped at the stopping time $\sigma(\tau)$.*

Consider the domain $V_M = \{Re(z) < M\}\backslash L$, where $L$ is the ray in $\mathbb{C}$ emanating from $2\pi i$ horizontally to the left; a picture of $V_M$ can be found in Fig. 1. We note that if $M$ is sufficiently large then $P_0(B_{T(V_M)} \in L) > 1 - \varepsilon$; this is evident, since as $M \nearrow \infty$ we have $P_0(B_{T(V_M)} \in L) \nearrow P_0(T(V_\infty) < \infty)$, where $V_\infty = \mathbb{C}\backslash L$,